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THE GUNN EFFECT: INSTABILITY OF THE STEADY STATE AND STABILITY OF THE SOLITARY WAVE IN LONG EXTRINSIC SEMICONDUCTORS*

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Abstract. A linear stability analysis of the stationary solution of a one-dimensional drift-diffusion model used to describe the Gunn effect in GaAs is performed. It is shown that for long semiconductor samples under dc voltage bias conditions, and small diffusivity, the steady state may lose stability via a Hopf bifurcation. In the limit of infinitely long samples, there is a quasicontinuum of oscillatory modes of the equation linearized about the steady state that a acquire positive real part for voltages larger than a certain critical value. The linear stability of the solitary wave characteristic of the Gunn effect is proved for an idealized electron velocity curve in the zero diffusion limit.

Key words. Gunn effect, current instabilities in semiconductors, Hopf bifurcation

AMS subject classifications. 35G25, 35M05, 65M15

1. Introduction. The Gunn effect is observed when a dc voltage is applied through a purely resistive circuit to a semiconductor displaying negative differential resistance (NDR) over a range of values of the electric field. When the dc voltage exceeds a certain threshold, a solitary wave is created at one of the two metallic contacts that limit the semiconductor, it propagates through the sample, and is destroyed at the other contact. This process, repeated periodically, gives rise to a periodic oscillation of the current in the external circuit which constitutes the Gunn effect. In experiments by Gunn, [11]–[13], the electric field was measured along a long negatively doped gallium arsenide (GaAs) semiconductor sample and periodic oscillations of the electric current in the microwave range were directly observed. An asymptotic analysis of the Gunn effect in the classical drift-diffusion model has been performed recently [14] (see also [1], [4], [18], and [19]), but there remain important points to be elucidated. The study of the onset of the Gunn effect is among these (see [5]), as experiments in n-GaAs [15] and p-Ge [16], [17], [20], [31] seem to indicate that interesting complex phenomena, including intermittency, may occur in the onset region. Another important open problem is proving the stability of the fully developed Gunn effect solutions that were constructed asymptotically in [14].

In this paper we show that the base state, corresponding to a time-independent electric field in the semiconductor, loses stability because complex conjugate eigenvalues of the linear stability problem cross the imaginary axis. This suggests that a Hopf

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bifurcation occurs, which will be confirmed elsewhere [5]. We also show that, in the limit of infinitely long semiconductors, a quasicontinuum of oscillatory modes becomes unstable as the voltage surpasses a critical value, something that was conjectured previously by one of us using a very simplified description of the steady as a two-step profile for a different model [2] (see also [6] and [7]). With respect to the stability of the Gunn oscillations constructed in [14], we only have a partial result: for an idealized shape of the electron velocity curve and ignoring diffusion, we are able to prove that the fully developed solitary wave (far from the contacts) is linearly stable. In the literature there are incomplete proofs of the stability of the solitary wave for the general case (general electron velocity and nonzero diffusion) [8], [32], all of which make a separation of variables ansatz (see the discussion in [3]). Our proof follows a different path and thus we include it here in spite of the drastic simplifications of the model we have made in the hope that it could lead to a proof for the general case.

To describe these phenomena, we use the following model:

$$\frac{\partial E}{\partial x} = n - 1, \tag{1.1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial J_e}{\partial x} = 0, \tag{1.2}$$

$$J_e = v(E)n - \delta \frac{\partial n}{\partial x}. \tag{1.3}$$

This model is a reduced form of the drift-diffusion equations [24], where the transport of holes has been neglected. Equation (1.1) is Poisson's law for (minus) the electric field $E(x,t)$ inside the sample. Equation (1.2) is the continuity equation for the density of the mobile carriers (electrons), $n(x,t)$, and J_e is the electron current per unit area of the sample cross section, given by (1.3). Equations (1.1)–(1.3) are written here in nondimensional form, [1], and they should be supplemented by the voltage bias condition (see below), and boundary and initial conditions. While more complicated models also display Gunn oscillations [9], [25], the present model is generally agreed to be sufficient to account for the experimental observations in semiconductors longer than 1 μm [29], [30].

In these equations the concentration of donor impurities has been assumed to be uniform and constant (and set equal to one) and δ ($0 < \delta \ll 1$) is a constant diffusion coefficient for the electrons [1]. (A field-dependent small diffusivity can be considered with minor changes.) $v(E)$ is the electron drift velocity, an N-shaped function with a region of negative slope reflecting the effect of the transfer of hot electrons between different valleys of the conduction band. It is the negative slope that gives rise to NDR [27], [28], [32]. The velocity function can be modelled as [21]:

$$v(E) = E\frac{1 + BE^4}{1 + E^4}, \qquad 0 < B < 0.36. \tag{1.4}$$

We eliminate the carrier concentration $n(x,t)$ using (1.1) in (1.2), and then integrate with respect to x, thereby finding

$$\frac{\partial E}{\partial t} + v(E)\left[\frac{\partial E}{\partial x} + 1\right] - \delta \frac{\partial^2 E}{\partial x^2} = J, \qquad 0 < x < L, \quad t > 0, \tag{1.5}$$

which is Ampere's law: the constant of integration $J = J(t)$ can be identified with the total current, sum of the displacement current, $\partial E/\partial t$, and of the electron current J_e.

The current $J(t)$ is essentially that which is measured on an purely resistive external circuit [1], [4]. Equation (1.5) and the voltage bias condition,

$$\int_0^L E(x,t)\,dx = \phi \tag{1.6}$$

are the equations for the unknowns $E(x,t)$ and $J(t)$. Ignoring local details at the metal-semiconductor region, we will assume Ohm's law at the two ends of the semiconductor: $E = \rho J_e$, or, by (1.3) and (1.5),

$$E + \rho\left(\frac{\partial E}{\partial t} - J\right) = 0, \quad \text{at } x = 0, L, \quad t > 0, \tag{1.7}$$

where $\rho \geq 0$ is the dimensionless resistivity of the contacts at $x = 0$ and $x = L$. Contacts with different resistivities at $x = 0, L$ can also be considered with obvious changes in our results. The idea of using a boundary condition at the contacts that locally relates the electron current with the field is due to Kroemer [22]. Contact current density-electric field relationships more general than the linear law (1.7) have been used by Grubin [10] (He considered a general monotone increasing curve; see also [28, p. 200]). For the purpose of the present paper, viz linear stability analyses of the steady state and of the solitary wave, a general curve will only introduce minor changes in our results. In particular one should replace ρ by the reciprocal of the slope of the current-field contact characteristic curve at the proper value of the field [2], [3]. Together with the boundary conditions (1.7) and a convenient initial condition,

$$E(x,0) = f(x) \geq 0, \qquad 0 < x < L. \tag{1.8}$$

Equations (1.5)–(1.8) constitute a mathematically well-posed problem (see global existence, uniqueness, and smoothness properties in [23]).

For most of the experiments performed on GaAs samples [11]–[13], the dimensionless length L is very large and the dimensionless diffusivity δ is very small. (Notice that a 10 micron GaAs sample was assumed in [1], which is quite short compared to Gunn's 200 micron samples in other experiments [12]–[13].) Here we thus consider the limits $\delta \downarrow 0$ and $L \to +\infty$.

In the limit $\delta \downarrow 0$ we can resort to matched asymptotic expansions to understand the Gunn effect. The positive sign of the convective term in (1.5) implies that disturbances propagate from left to right. This in turn indicates that a boundary layer appears at the receiving contact $x = L$ [1], [2]. Setting $\delta = 0$ in (1.5) and ignoring the boundary condition at $x = L$, we obtain the reduced outer equation

$$\frac{\partial E}{\partial t} + v(E)\left[\frac{\partial E}{\partial x} + 1\right] = J, \qquad 0 < x < L, \quad t > 0, \tag{1.9}$$

to be solved together with (1.6) and appropriate initial data (1.8) and the boundary condition

$$E(0,t) + \rho\left[\frac{\partial E(0,t)}{\partial t} - J(t)\right] = 0, \qquad t > 0. \tag{1.10}$$

Near $x = L$, we must insert a diffusive quasistationary boundary layer [1], [2], with

$$v(E)\frac{\partial E}{\partial x} \sim \delta\frac{\partial^2 E}{\partial x^2}, \tag{1.11}$$

$$E(L,t) = E_c(t), \quad \text{with } \frac{dE_c}{dt} + \frac{E_c}{\rho} = J(t), \tag{1.12}$$

$$E(x,t) \sim E_{\text{out}} \quad \text{as } (L-x) \gg \delta \quad (\delta \downarrow 0). \tag{1.13}$$

Equation (1.12) is the boundary condition at $x = L$ and (1.13) is the matching condition with the solution of the outer problem; E_{out} is used here to denote the outer solution at $x = L$. The solution of (1.11)–(1.13) is

$$\frac{\partial E}{\partial \xi} = Q(E, E_{\text{out}}(L,t)), \qquad \xi \equiv \frac{x-L}{\delta}, \tag{1.14}$$

$$Q(E, E_{\text{out}}) = \int_{E_{\text{out}}}^{E} v(s)\,ds, \tag{1.15}$$

$$\xi = \int_{E_c(t)}^{E(\xi,t)} \frac{dE'}{Q(E', E_{\text{out}})}. \tag{1.16}$$

In writing down the quasistationary boundary layer problem (1.11), we have used the following argument. The natural time scale for the evolution of the boundary layer solution is $\tau = t/\delta$. At this fast time scale the outer solution is quasistationary (it does not have time to change) and the normal modes with eigenvalues of order $1/\delta$ associated with the evolution of a small disturbance about (1.14)–(1.16) are all stable, as indicated later in Appendix B. Thus no instability appears at the boundary layer at these short time scales and (1.11) is a reasonable scaling for slower processes.

The rest of this paper is organized as follows. In §2 we construct the steady state of the problem (1.6), (1.9), and (1.10) using matched asymptotic expansions in the limit $L \to +\infty$. The field at the diffusive layer is given by (1.16) and it can be used straightforwardly to build a uniform approximation of the steady state of the complete (1.5)–(1.7). Then in § 3 we study its linear stability in the relevant limit $\delta \downarrow 0$ and $L \to +\infty$ (with $L\delta \downarrow 0$), and we show that the steady state loses stability via a Hopf bifurcation. Section 4 contains a brief reminder of the asymptotic construction of the solitary wave far from the contacts (in the limit $\delta \downarrow 0$ and $L \to +\infty$ with $L\delta \downarrow 0$) and our linear stability proof for a very simple $v(E)$ curve and $\delta = 0$. Lastly, §5 is devoted to a discussion of our results, and several technical matters are considered in the Appendices.

2. The steady state. When E and J are time-independent, (1.9) and (1.10) become

$$\frac{\partial E}{\partial x} = \frac{J - v(E)}{v(E)}, \tag{2.1}$$

$$E(0) = \rho J. \tag{2.2}$$

Let $E(x;J)$ denote the solution of (2.1)–(2.2) corresponding to a given positive J. The current J is then determined as a function of the applied voltage ϕ by the equation

$$\Phi(J) \equiv \int_0^L E(x;J)\,dx = \phi, \tag{2.3}$$

which has a unique solution J for every positive ϕ. To see this we observe the following.

(a) The function $\Phi(J)$ is monotone increasing. Indeed, let $F(x;J)=\partial_J E(x;J)$. By differentiating (2.1) with respect to J we obtain easily

$$v(E)\frac{\partial F}{\partial x}+v'(E)\left(1+\frac{\partial E}{\partial x}\right)F=1, \quad F(0;J)=\rho>0. \tag{2.4}$$

Integrating this first-order equation for F we verify directly that $F=\partial_J E(x;J)>0$.

(b) $\lim_{J\downarrow 0}\Phi(J)=0$.

Indeed, $E(0;J)=\rho J\to 0$ as $J\to 0$. Furthermore, for small J, (2.1) has a unique fixed point $E_1(J)$ which is a global attractor as x increases and which tends to zero as $J\to 0$. Thus for each $x, \lim_{J\downarrow 0}E(x;J)=0$, and $\lim_{J\downarrow 0}\Phi(J)=0$.

(c) $\lim_{J\downarrow\infty}\Phi(J)=\infty$, by a similar argument.

The qualitative behavior of $E(x;J)$ and of $\Phi(J)$ are seen by the analysis of the one-dimensional phase diagram of (2.1). Its fixed points satisfy $J-v(E)=0$. When the curve $v=v(E)$ is as in Fig. 1 and $v_m<J<v_M$, the function $J-v(E)$ has three zeros $E_1(J)<E_2(J)<E_3(J)$. The point E_2 is unstable, while E_1 and E_3 are attractors with basins $E<E_2$ and $E>E_2$, respectively. The asymptotic value of $E(x;J)$ as x increases (in long enough samples) is $E_1(J)$ or $E_3(J)$ depending on whether the initial point (E,J) at $x=0$ (which lies on the line $E=\rho J$) is to the left or to the right of $E_2(J)$, respectively. Of particular interest is the value $J=J_c$ for which these two coincide:

$$E_2(J_c)=\rho J_c. \tag{2.5}$$

This relation holds only for resistivities $\rho\in(E_M/v_M, E_m/v_m)$, which we will assume is the case in the rest of this paper. It is known that the Gunn effect is then mediated by solitary wave dynamics [14]. For smaller ρ, the Gunn effect is mediated by monopole

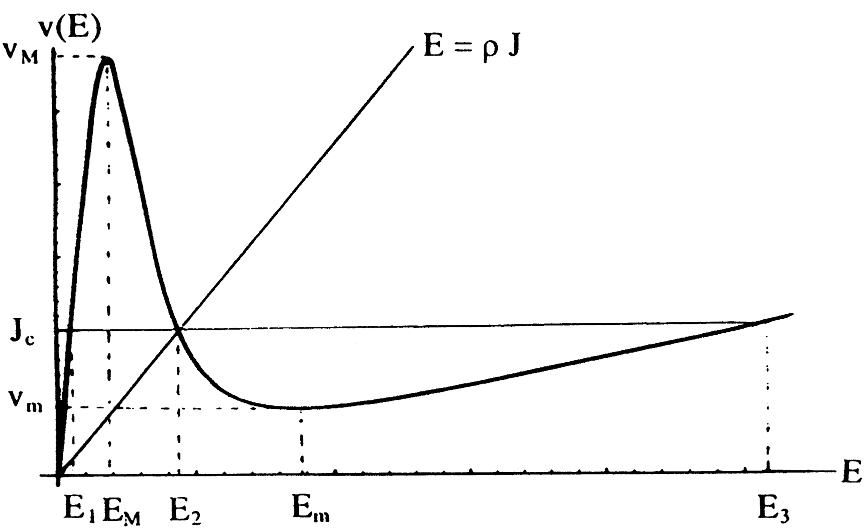

FIG. 1. *Electron velocity versus electric field* (1.4 *with* $B=0.02$. $v_M=v(E_M)$, $v_m=v(E_m)$.

wavefronts (see [14]), while for $\rho > E_m/v_m$ the present model is probably unrealistic and it has not been studied.

Let us determine the dependence of the current J with the average field ϕ/L for the problem (2.1)–(2.3) in the limit $L \to +\infty$. For $0 < J < J_c$, the field of the steady state decreases from $E(0; J) = \rho J$ down towards its asymptotic value $E_1(J)$; whereas for $J > J_c$, $E(x; J)$ monotonically increases from $E(0; J) = \rho J$ up to $E_3(J)$ for large x (see Fig. 2). In the limit $L \to +\infty$, the voltage $\Phi(J)$ of (2.3) is then approximately to

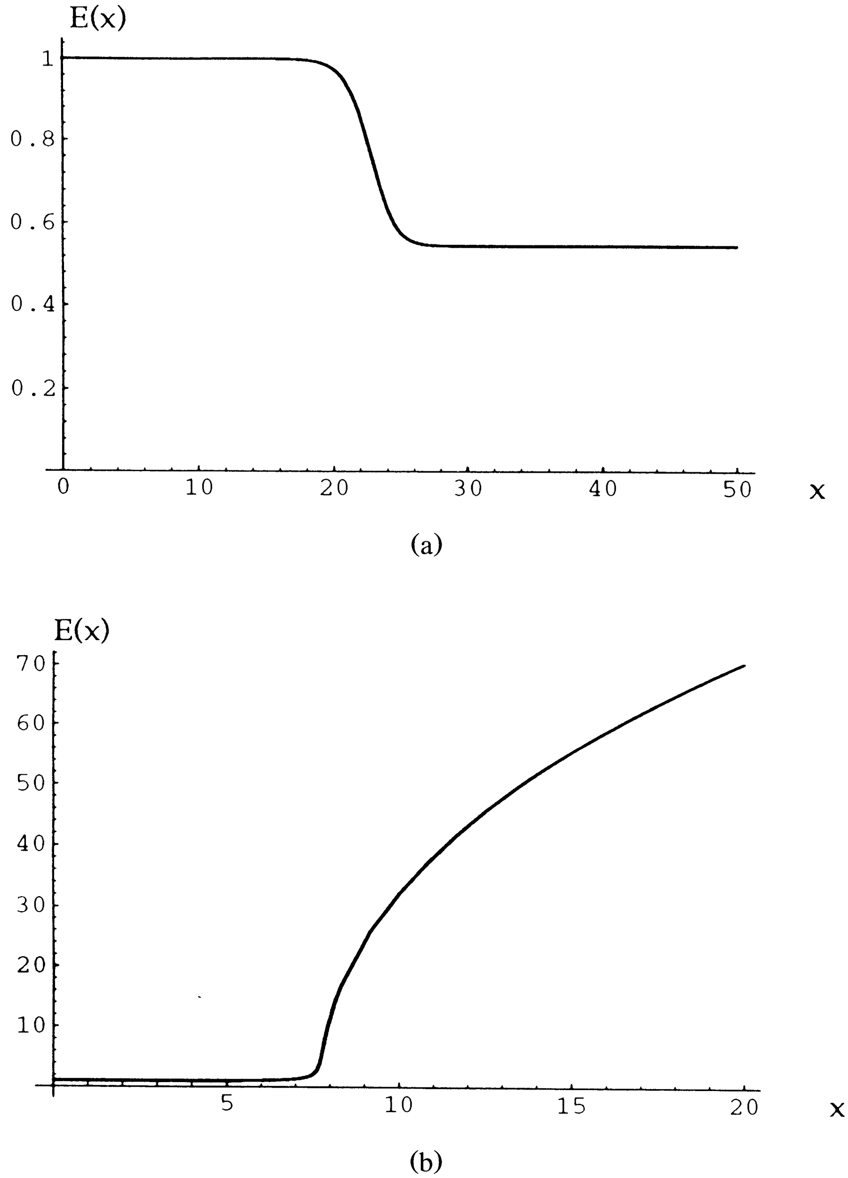

FIG. 2. *Steady state $E(x; J)$ when*: (a) $E_1(J_c)L < \phi < E_2(J_c))L$, $J \uparrow J_c$ *(numerical values*: $J_c - J = 5.01 \times 10^{-4}$; $\rho = 2$); (b) $E_2(J_c)L < \phi < E_3(J_c)L$, $J \downarrow J_c$ *(numerical values*: $J - J_c = 10^{-4}$; $\rho = 2$). *We have chosen a large enough* L *and the narrow boundary layer at* $x = L$ *has been omitted.*

leading order $E_1(J)L$ for $0<J<J_c$, and $E_3(J)L$ for $J_c<J$. We then have that

$$J \sim v\left(\frac{\phi}{L}\right), \tag{2.6}$$

both for $0<\phi/L<E_1(J_c)$ and for $E_3(J_c)<\phi/L$, asymptotically as $L\to+\infty$. The steady-state current follows therefore the curve $v(E)$ outside a voltage interval such that $E_1(J_c)<\phi/L<E_3(J_c)$. Inside this interval, $J\sim J_c$. We will now describe the steady-state field corresponding to such voltages.

Clearly, when $|J-J_c|\ll 1$ we have $E(0;J)\sim E_2(J_c)$. For what length X measured from $x=0$ does $E(x;J)$ stay close to the unstable value $E_2(J_c)$? Since the transition from E_2 to E_1 or E_3 occurs over a layer of order one thickness in the x-scale, we have

$$\phi \sim E_j(J_c)(L-X)+E_2(J_c)X, \qquad j=1,3 \quad (L\gg 1), \tag{2.7}$$

from which we deduce

$$X\sim \frac{\phi-E_1(J_c)L}{E_2(J_c)-E_1(J_c)}, \quad \text{when } J<J_c \quad [\text{hence } \phi<E_2(J_c)L], \tag{2.8a}$$

$$X\sim \frac{E_3(J_c)L-\phi}{E_3(J_c)-E_2(J_c)}, \quad \text{when } J>J_c \quad [\text{hence } \phi>E_2(J_c)L]. \tag{2.8b}$$

More precise calculations are carried out in Appendix A. The main results are contained in the following.

LEMMA 1. (a) *Let $E_1(J_c)<\phi/L<E_2(J_c)$. Let us fix $E(X;J)=E_0$, where E_0 is a given number in the interval $E_1(J_c)<E_0<E_2(J_c)$. Then for each value of $X\in(0,L)$, such that $\{X,(L-X)\}\gg 1$, ϕ and J are uniquely determined by the asymptotic formulae*:

$$\phi=E_1L+(E_2-E_1)X+\left(\frac{L-X}{v_1'}+\frac{X}{v_2'}\right)(J-J_c)+\int_{E_2}^{E_0}\frac{(s-E_2)v(s)}{J_c-v(s)}\,ds + \int_{E_0}^{E_1}\frac{(s-E_1)v(s)}{J_c-v(s)}\,ds+o(1), \tag{2.9a}$$

and

$$J-J_c\sim -\frac{c_L|v_2'|}{1-\rho v_2'}\exp\left\{-\frac{|v_2'|X}{J_c}\right\}. \tag{2.9b}$$

Also we have the asymptotic relations

$$\begin{aligned} &E(x;J)\sim E_2-c_L\exp\left\{-\frac{v_2'(x-X)}{J}\right\} \quad (\text{as } (x-X)\to-\infty),\\ &\text{with}\quad c_L=(E_2-E_0)\exp\left\{\int_{E_0}^{E_2}\left[\frac{1}{s-E_2}-\frac{v_2'}{v(s)-J}-\frac{v_2'}{J}\right]ds\right\}, \end{aligned} \tag{2.10a}$$

$$\begin{aligned} &E(x;J)\sim E_1+c_R\exp\left\{-\frac{v_1'(x-X)}{J}\right\} \quad (\text{as } (x-X)\to+\infty),\\ &\text{with}\quad c_R=(E_0-E_1)\exp\left\{-\int_{E_1}^{E_0}\left[\frac{1}{s-E_1}-\frac{v_1'}{v(s)-J}-\frac{v_1'}{J}\right]ds\right\}. \end{aligned} \tag{2.10b}$$

In the right-hand sides of (2.9a) *and* (2.9b), $v_k' \equiv v'(E_k(J))$, $[v'(E) \equiv dv/dE]$, $k = 1, 2$. $E(x; J)$ *in* (2.10) *refers to the field in the transition layer*: $(x - X) \to -\infty$ (*resp*., $+\infty$) *means leaving the transition layer towards the left* (*resp*., *right*). *Notice that* c_L *and* c_R *are both positive*.

(b) *Let* $E_2(J_c) < \phi/L < E_3(J_c)$. *Let us fix* $E(X; J) = E_0$, *where* E_0 *is a given number in the interval* $E_2(J_c) < E_0 < E_3(J_c)$. *Then for each value of* $X \in (0, L)$, *such that* $\{X, (L - X)\} \gg 1$, ϕ *and* J *are uniquely determined by the asymptotic formulae* (2.9)–(2.10) *where* $E_3(J)$ *replaces* $E_1(J)$ *everywhere*. *Notice that* c_L *and* c_R *are now both negative*.

Proof. See Appendix A.

Remark 1. $X = O(\ln L)$ for voltages near the instability threshold (see §3). Then the first term on the right-hand side of (2.9a) is asymptotic to $(J - J_c)L/v_1'$, of the same order, $O(1)$, as the other two terms on this equation.

3. Linear stability of the steady state. In this section we study the linear stability of the steady state. We formulate the problem and find numerically the region in the $(\phi/L, L)$ plane where the steady state is linearly unstable (Fig. 3). We shall see that the steady state may become linearly unstable only for L larger than a certain minimal length L_m which is near 1. For each $L > L_m$, the steady state is linearly unstable when $\phi \in (\phi_\alpha, \phi_\omega)$. As L increases, ϕ_α/L rapidly approaches $E_1(J_c)$. This suggests that the $L \to +\infty$ approximations constructed in §2 could be of rather extensive practical use for ϕ near ϕ_α (for ϕ near ϕ_ω, our asymptotic results will be good approximations only for unrealistically long semiconductors). We therefore find

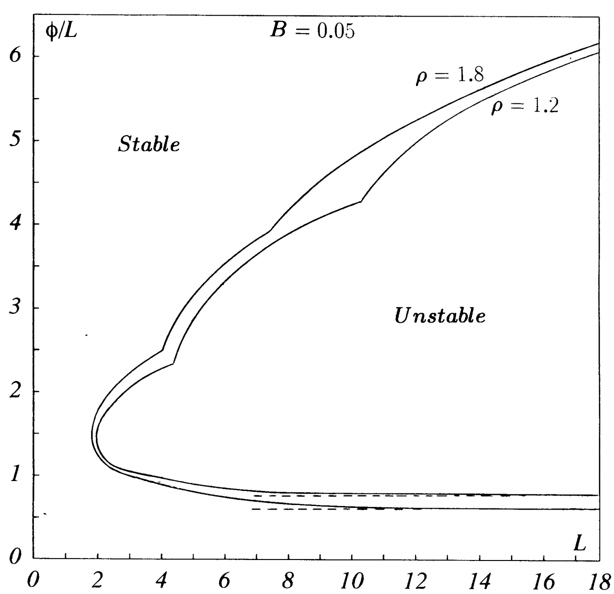

FIG. 3. *Neutral stability curve of the steady state for* $B = 0.05$ *and two values of* ρ. *The steady state may be linearly unstable to the right of the minimal length* L_m. *Notice that* L_m *decreases with the resistivity of the injecting contact* ρ. *For each* $L > L_m$, *the steady state is linearly stable outside a voltage interval* $(\phi_\alpha, \phi_\omega)$: ϕ_α *corresponds to the lower branch of the neutral stability curve and* ϕ_ω *to the upper branch. The dotted lines indicate the value* $E_1(J_c)$ *to which the lower branch of the neutral stability curve tends as* $L \to +\infty$. *The upper branch of the neutral stability curve tends to* $E_3(J_c)$ *as* $L \to +\infty$, *but a good approximation to this asymptotic value occurs only for much larger values of* L *than those represented in the figure*.

the critical values of ϕ_α/L and of ϕ_ω/L, as $L \to +\infty$ and describe how the steady state becomes linearly unstable at them.

Let

$$J(t) = J + \varepsilon\tilde{j}(t), \qquad E(x,t) = E(x) + \varepsilon\tilde{e}(x,t) \quad 0 < \varepsilon \ll 1, \tag{3.1}$$

be a disturbance about the steady state. Inserting (3.1) into (1.6), (1.9), and (1.10) yields

$$\frac{\partial \tilde{e}}{\partial t} + v(E)\frac{\partial \tilde{e}}{\partial x} + \left(1 + \frac{\partial E}{\partial x}\right)v'(E)\tilde{e} - \tilde{j}(t) = 0, \tag{3.2a}$$

$$\tilde{e}(0,t) + \rho\left[\frac{\partial \tilde{e}(0,t)}{\partial t} - \tilde{j}(t)\right] = 0, \tag{3.2b}$$

$$\int_0^L \tilde{e}(x,t)\,dx = 0. \tag{3.2c}$$

That considering a small nonzero diffusivity $0 < \delta \ll 1$ does not modify our stability results is shown in Appendix B.

Equations (3.2) can be solved by the separation of variables:

$$\tilde{j}(t) = je^{\lambda t}, \qquad \tilde{e}(x,t) = e^{\lambda t}\hat{e}(x;\lambda). \tag{3.3}$$

Insertion of (3.3) in (3.2) yields

$$\frac{\partial[v(E)\hat{e}]}{\partial x} + \{\lambda + v'(E)\}\hat{e} = j, \tag{3.4a}$$

$$Z(\lambda) \equiv \int_0^L \frac{\hat{e}(x;\lambda)}{j}\,dx = 0. \tag{3.4b}$$

$$\hat{e}(0;\lambda) = \frac{\rho j}{1 + \rho\lambda}. \tag{3.4c}$$

The zero of the impedance $Z(\lambda)$ with largest real part determines the linear stability of the steady state. We have evaluated numerically the neutral stability curve (corresponding to the zero with largest real part being pure imaginary) for the steady state in the parameter space ϕ/L vs. L for different values of the resistivity ρ. The results are shown in Fig. 3. The discontinuities in the slope of the curve in Fig. 3 are due to the crossing of different zeroes as L increases. Notice that above a certain $L = L_m$, there are two values of the voltage for each L, ϕ_α and ϕ_ω, such that the steady state is linearly stable for ϕ outside $(\phi_\alpha, \phi_\omega)$. Notice also that ϕ_α/L rapidly tends to a constant value $E_1(J_c)$ as L increases. This suggests that the results we will obtain next in the asymptotic limit $L \to +\infty$ may be of practical applicability even for moderate L.

For long semiconductors the linear stability of the steady state can be ascertained without resorting to numerical calculations. For voltages smaller than $E_1(J_c)L$ or larger than $E_3(J_c)L$, the steady state is linearly stable [28], [4]. Considering the steady-state profile for $J \sim J_c$ (which corresponds to all other voltages, as said in the previous section), we can distinguish regions to the left and right of $x = X$ where the coefficients of (3.4a) are basically constant ($E_2(J_c)$ and $E_1(J_c)$) and a transition region connecting them. Let us restrict our attention to the eigenvalue with largest real part and look for the instability threshold. We shall assume that $\operatorname{Re}\lambda + v_1' > 0$, for otherwise $\operatorname{Re}\lambda < 0$, the steady state is linearly stable and we are not near the instability threshold. Similarly, we shall assume that $\operatorname{Re}\lambda + v_2' < 0$, for otherwise $\operatorname{Re}\lambda > 0$, the steady state is linearly unstable, and we have missed again the instability threshold. Thus $v_2' < -\operatorname{Re}\lambda < v_1'$. We make the asymptotic ansatz that $1 \ll X \ll L$ at the neutral stability limit. The ansatz will be verified a posteriori. Then a large contribution to the impedance (3.4b) will come from the solution to the right of $x = X$, whose width is $(L - X) \sim L$. In an $O(1)$ distance after $x = X$, $\hat{e}(x;\lambda)$ exponentially decreases to the constant solution of (3.4a) with $E = E_1(J_c)$, so that

$$\int_{X+\Delta x}^{L} \frac{\hat{e}(x;\lambda)}{j}\,dx \sim \frac{L}{v_1' + \lambda}, \tag{3.5}$$

where $1 \ll \Delta x \ll X$ is arbitrary. To obtain the leading order approximation to the impedance $Z(\lambda)$ we need to find the dominant contribution of the other regions. The transition region extends to the left up to $x = 0$, while to the right $E(x; J_c)$ rapidly decreases to $E = E_1(J_c)$. We shall write the solution of (3.4a) and (3.4c) there as

$$\hat{e}(x;\lambda) = j\left\{\hat{e}_p(x;\lambda) + h(x;\lambda)\frac{\partial E(x;J)}{\partial x}\right\}, \tag{3.6}$$

where $\hat{e}_p(x;\lambda)$ obeys (3.4a) with the following boundary condition:

$$\hat{e}_p(0;\lambda) = \frac{1}{v_2' + \lambda} \sim \hat{e}_p(X - \Delta x;\lambda) \quad (\text{as } \Delta x \to +\infty). \tag{3.7a}$$

The condition (3.7b) below is automatically satisfied by any solution of the first-order equation (3.4a) when $\operatorname{Re}\lambda + v_1' > 0$:

$$\hat{e}_p(X + \Delta x;\lambda) \sim \frac{1}{v_1' + \lambda} \quad (\text{as } \Delta x \to +\infty). \tag{3.7b}$$

Inserting (3.6) into (3.4a) and (3.4c) and taking (3.7) into account, we find that $h(x;\lambda)$ is the solution of

$$v(E)\frac{\partial h}{\partial x} + \lambda h = 0, \tag{3.8a}$$

$$h(0;\lambda)\frac{\partial E(0;J)}{\partial x} = c_0 \equiv \frac{1 - \rho v_2'}{(1 + \rho\lambda)(-\lambda - v_2')}, \tag{3.8b}$$

where

$$\frac{\partial E(0;J)}{\partial x} \sim \frac{c_L v_2'}{J}\exp\left(\frac{v_2' X}{J}\right) \tag{3.8c}$$

from (2.10a). The general solution of (3.8a) is

$$h(x;\lambda) = h(X;\lambda)\exp\left(-\frac{\lambda}{J}[x - X + E(x;J) - E_0]\right), \tag{3.9}$$

where (2.1) has been used. We calculate $h(X;\lambda)$ by inserting (3.9) in the boundary condition (3.8b):

$$h(X;\lambda) \sim \frac{c_0 J}{c_L v_2'}\exp\left(-\frac{(v_2' + \lambda)X + \lambda(E_0 - E_2)}{J}\right). \tag{3.10}$$

Notice now that, whenever $v_2' + \mathrm{Re}\,\lambda < 0$, the large exponential factor in (3.10) makes the second term in (3.6) dominant over the first one in the transition region. Then the contribution of (3.6) to the impedance can be approximated by

$$\text{(3.11)}\quad \int_0^{X+\Delta x} \frac{\hat{e}(x;\lambda)}{j}\,dx \sim h(X;\lambda) \times \int_0^{X+\Delta x} \frac{\partial E}{\partial x} \exp\left\{-\frac{\lambda}{J}[x - X + E(x;J) - E_0]\right\} dx.$$

Adding (3.5) and (3.11) we obtain

$$\text{(3.12)}\qquad Z(\lambda) \sim \frac{L}{v_1' + \lambda} + \frac{c_0 J}{c_L v_2'} K(\lambda) \exp\left\{-\frac{(v_2' + \lambda)X}{J}\right\},$$

$$\text{(3.13)}\qquad K(\lambda) = \int_0^{X+\Delta x} \frac{\partial E}{\partial x} \exp\left\{-\frac{\lambda}{J}[x - X + E(x;J) - E_2]\right\} dx.$$

Notice that (3.12) holds for any value of the eigenvalue λ such that $v_2' < -\mathrm{Re}\,\lambda < v_1'$. (2.10) then implies that the integrand in (3.13) tends exponentially to zero as $|x - X| \to \infty$. This means that we can replace the endpoints of the integral (3.13) by $\pm\infty$. At the instability threshold, $\lambda = i\Omega$, and $|K(i\Omega)| < |K(0)| = E_2 - E_1$ if Ω is a nonzero real. Thus $|K(i\Omega)|$ is of order 1 and (3.12) implies that $\mathrm{Re}\,\lambda = 0$ at $X \sim X_c(\Omega)$ given by

$$\text{(3.14a)}\qquad X_c(\Omega) = \frac{J}{|v_2'|} \ln\left\{\frac{c_L L |v_2'(v_2' + i\Omega)(1 + i\rho\Omega)|}{J(1 - \rho v_2')|(v_1' + i\Omega)K(i\Omega)|}\right\},$$

(3.14b)

$$\frac{\Omega X_c(\Omega)}{J} \sim 2n\pi + \arg\left\{\frac{(v_1' + i\Omega)K(i\Omega)}{(|v_2'| - i\Omega)(1 + i\rho\Omega)}\right\}, \quad n = 0, \pm 1, \pm 2, \ldots.$$

It is straightforward to prove that $X_c(\Omega)$ in (3.14a) is an increasing function. Let us consider ϕ close to ϕ_α for which (2.9a) holds. Since the voltage is an increasing function of X (cf. (2.9a)) and we know that the steady state is unstable for $\phi > \phi_\alpha$, the instability threshold corresponds to the lowest possible Ω. Let us now consider ϕ close to ϕ_ω for which (2.9a) holds if E_3 replaces E_1 in it. Now, the area under the electric field (numerically equal to ϕ) is a decreasing function of X, and therefore of Ω. We know that the steady state is unstable for $\phi < \phi_\omega$, and therefore ϕ_ω corresponds to the lowest possible Ω, as was also the case of ϕ_α. From (3.14a) and (3.14b) we thus see that the instability threshold is reached at $X = X_c \sim X_c(0)$,

$$\text{(3.15)}\qquad X_c \sim \frac{J}{|v_2'|} \ln\left\{\frac{c_L v_2'^2 L}{J v_1'(1 - \rho v_2')(E_2 - E_1)}\right\} = \frac{J}{|v_2'|} \ln L + O(1).$$

Remark 2. Condition (3.15) has been obtained assuming $\mathrm{Re}\,\lambda + v_2' < 0$. We may wonder whether it is not possible to find unstable eigenvalues outside this range so that the steady state be unstable for X smaller than (3.15). The answer is no: If $\mathrm{Re}\,\lambda + v_2' > 0$, (3.5) still holds, but now the contribution of the high field region $x \in (0, X + \Delta x)$ to the impedance is (at most, for large enough X and/or $\mathrm{Re}\,\lambda$),

$X/(v_2' + \lambda) = O(X/\lambda)$ which, for $X < X_c$, cannot cancel the $O(L/\lambda)$ term (3.5). Thus $Z(\lambda)$ does not have zeros with $\operatorname{Re}\lambda > |v_2'|$ for $X < X_c$, and the steady state is then always linearly stable for such X.

Condition (3.15) yields asymptotic estimations of the critical voltages: ϕ_α via (2.9a) and of ϕ_ω via (2.9a) with E_3 instead of E_1. In both cases at $X = X_c$ the eigenvalues with largest real part are $\lambda_n = i\Omega_n[1 + o(1)]$,

$$\Omega_n \sim \frac{(2n+1)J\pi}{X_c}, \qquad n = 0, \pm 1, \pm 2, \ldots, o(\ln L). \tag{3.16}$$

Each eigenvalue λ_n has zero real part at $X = X_c(\Omega_n) > X_c(\Omega_{n-1})$. These values of $X_c(\Omega)$ are asymptotic to $X_c(0) \sim X_c$ up to $o(1)$ terms. Then at $X \sim X_c, |\lambda_n - i\Omega_n| \ll \Omega_n$. At $X = X_c$, all the eigenvalues have negative real parts except for $\lambda_0 = i\Omega_0$. For slightly larger X (or, equivalently, voltages slightly inside the unstable region in Fig. 3), more and more eigenvalues acquire positive real parts. We will now calculate the relationship between the $o(\Omega_n)$-corrections to each eigenvalue, $\lambda_n - i\Omega_n$, and $(X - X_c), \Omega_n$.

For voltages $\phi \in (\phi_\alpha, \phi_\omega)$, X is larger than (3.15) which means that $\operatorname{Re}\lambda \neq 0$ (in fact, $\operatorname{Re}\lambda > 0$ by continuity). Near the critical voltage $\lambda = o(1)$ according to (3.16). Then we may expand c_0 in (3.12) and the integrand in (3.13) in powers of λ, and the condition $Z(\lambda) = 0$ becomes

$$-1 \sim \frac{(E_2 - E_1)(1 - \rho v_2')Jv_1'}{c_L v_2'^2} F(\lambda) \exp\left\{ -\frac{(v_2' + \lambda)X}{J} - \ln L \right\}, \tag{3.17}$$

$$F(\lambda) = 1 + f_1\lambda + f_2\lambda^2 + O(\lambda^3), \tag{3.18a}$$

$$f_1 = -\rho - \frac{v_1' - v_2'}{v_1' v_2'} - \varphi_1, \tag{3.18b}$$

$$f_2 = (\rho + \varphi_1)\left(\rho + \frac{v_1' - v_2'}{v_1' v_2'}\right) + \frac{v_1' - v_2'}{v_1' v_2'^2} - \varphi_2. \tag{3.18c}$$

Here the φ_k's are the coefficients of the expression of the integral in the right-hand side of (3.13) (divided by $-K(0)$) in powers of λ:

$$\varphi_k = \frac{(-1)^k}{k!(E_2 - E_1)} \int_{-\infty}^{+\infty} \frac{\partial E(x;J)}{\partial x} \left(\frac{x - X + E(x;J) - E_2}{J} \right)^k dx, \qquad k = 1, 2, \ldots. \tag{3.19}$$

We have set $\pm\infty$ as the endpoints of the integral which only adds negligible exponentially small errors to (3.19). We now find the dispersion relation between $\operatorname{Re}\lambda, (\phi - \phi_\alpha)$ and $\operatorname{Im}\lambda$ near $X = X_c$ (equivalently, $\phi = \phi_\alpha$). This relation will demonstrate that $\partial \operatorname{Re}\lambda / \partial\phi > 0$ at $\phi = \phi_\alpha$ and, therefore, that the steady state may lose stability via a Hopf bifurcation, [5]. The following result relates changes in the voltage to changes in the location of the transition layer of the steady state:

LEMMA 2. *Let $X + X_c + \delta X$, $\delta X \ll 1$. Then the voltage corresponding to X is either*

$$\phi = \phi_\alpha + \delta\phi, \quad \text{with } \delta\phi \sim 2(E_2 - E_1)\delta X, \tag{3.20a}$$

or

$$\phi = \phi_\omega - \delta\phi, \quad \text{with } \delta\phi \sim 2(E_3 - E_2)\delta X. \tag{3.20b}$$

Proof. See Appendix C.

When $\phi = \phi_\alpha + \delta\phi$ (resp. $\phi = \phi_\omega - \delta\phi$), $\delta\phi \ll 1$, we have

$$\lambda = i\Omega_n + \alpha_n + i\omega_n, \qquad (\alpha_n, \omega_n) \ll \Omega_n \ll 1. \tag{3.21}$$

Here Ω_n is given by (3.16) and $\alpha_n = \omega_n = 0$ whenever $\phi = \phi_\alpha$ or $\phi = \phi_\omega$, so that α_n and ω_n tend to zero as $\delta\phi$ or δX tend to zero. As shown in Appendix C, the variation δJ of the steady-state current due to δX is of order $\delta X/L$, so that $\delta J \ll X_c \delta J \ll \delta X$ and we will ignore the variations of J due to δX in what follows. We now substitute (3.21) and (3.16) in (3.17) with $X = X_c + \delta X$, use (3.15) and (3.18a), and finally equate real and imaginary parts of the resulting expression. We find

$$\alpha_n \sim \frac{|v_2'|\delta X}{X_c} - \frac{J\left(f_2 - \frac{1}{2}f_1^2\right)\Omega_n^2}{X_c}, \tag{3.22a}$$

$$\omega_n \sim \frac{Jf_1\Omega_n}{X_c}. \tag{3.22b}$$

We have evaluated numerically the coefficient of Ω_n^2 in (3.22a) (which is *minus* an effective diffusivity since it multiplies the square of the wavevector Ω_n in the dispersion relation) for values of the resistivity in the interval $E_M/v_M < \rho < E_m/v_m$, with typical values of B ($B = 0.002$). The result is that the effective diffusivity (which is defined as *minus* the coefficient of Ω_n^2 in (3.22a)) is always positive.

δX is related to $\delta\phi$ by either (3.20a) or (3.20b). Consistency requires $\alpha_n \ll \Omega_n$, which implies $\Omega_n \ll X_c = O(\ln L)$ as stated in (3.16). Notice that, for fixed δX, the separation between the real parts of two consecutive eigenvalues decays to zero faster than $1/(\ln L)^2$:

$$\alpha_n - \alpha_{n-1} \sim \frac{8nJ^3\pi^2\left(f_2 - \frac{1}{2}f_1^2\right)}{X_c^3} = nO\left(\frac{1}{(\ln L)^3}\right) \ll \frac{1}{(\ln L)^2}. \tag{3.23}$$

From the dispersion relation (3.22a), and given (3.20), we see that many modes become unstable as $0 < \delta\phi \ll 1$ if

$$\frac{1}{(\ln L)^2} \ll \delta\phi \ll 1 \qquad (\ln L \to +\infty); \tag{3.24a}$$

the number of unstable modes is (cf. (3.22a) with $\alpha_N = 0$):

$$N = O\left(\sqrt{\delta\phi}\,\ln L\right). \tag{3.24b}$$

For voltages $\phi = \phi_\alpha + \delta\phi$ or $\phi = \phi_\omega - \delta\phi$, with $\delta\phi$ in the range (3.24a), a quasicontinuum of eigenvalues with vanishing frequencies (3.16) crosses the imaginary axis. The analysis of the resulting bifurcation will be presented elsewhere [5].

4. The solitary wave solution and its stability in an idealized case. Experiments [11]–[13], [28] indicate that in a range of voltages where the steady state is unstable, a pulse is generated at $x = 0$ and travels along the semiconductor. The pulse is destroyed at $x = L$, and the phenomenon repeated as another pulse is then created at $x = 0$. When the pulse is away from the endpoints the current is constant, and we can treat the pulse as a solitary wave riding at constant speed on a steady plateaux [18], [1], [14]. Two of us have constructed asymptotically the periodic Gunn oscillation for long semiconductors in a previous publication [14]. To prove the linear stability of the oscillation we should perform a Floquet-type analysis of the evolution problem linearized about the time-periodic solution. Instead of doing this, we try to reach a more modest goal: we focus our attention in the stage of a period of the Gunn oscillation that lasts longest, i.e., when there is a detached solitary wave that moves with constant current on a steady background field [14]. We try to prove that a small-amplitude perturbation of this situation decays in time before the wave reaches $x = L$. Even this more modest goal will prove elusive, as we shall see below.

There are four types of solitary wave solutions of (1.9) that correspond to solitary waves of (1.5) in the limit $\delta \downarrow 0$ [18].

(a) A triangular bump with a front having slope -1 and a shock in the tail.

(b) A trapezoidal pulse with front and back as in (a).

(c) A triangular depression with a shock in the front and a slope -1 at the tail.

(d) A trapezoidal depression with front and back as in (c). In the physics literature, pulses that rise above a baseline are called high field domains while depressions are called low field domains.

The solitary waves of (1.5) are constructed in [1] by the asymptotic matching of elementary solutions of (1.9) with constant J via intermediate boundary layer solutions of (1.5). In the case of a triangular high field domain the relevant solutions of (1.9) are (the other cases are similar and we will not consider them from now on):

(i) The steady state solution $E = E_1(J)$ outside the pulse;

(ii) The exact solution $E = Jt - x +$ const. for the front of the pulse;

(iii) The traveling shock solution at the back of the pulse. The velocity of a shock, $V(E_+, E_-)$, as a function of the values of the field on its right and left, E_+ and E_- respectively with $E_+ > E_-$, is [1], [18]

$$V(E_+, E_-) = \frac{\int_{E_-}^{E_+} v(E)\, dE}{E_+ - E_-}. \tag{4.1}$$

Clearly we have $E_- = E_1(J)$ and E_+ should satisfy $V(E_+, E_1(J)) = J$ for the pulse to move rigidly. J is determined from the voltage condition (1.6):

$$\phi \sim E_1(J)L + \tfrac{1}{2}[E_+(J) - E_1(J)]^2. \tag{4.2}$$

In the limit $\phi, L \to \infty, B \downarrow 0$ with $\phi = O(L)$ and $B\phi < \pi/4$, we have a triangular solitary wave with $E_1(J) \sim J \ll 1$ [14] so that (4.2) becomes

$$\phi \sim \tfrac{1}{2}E_+^2; \quad \text{therefore } E_+ \sim \sqrt{2\phi}\,. \tag{4.3}$$

J can now be calculated from (4.1) and (4.3), and the result confirms that we can ignore the first term in the right-hand side of (4.2) when deriving (4.3) [14].

We now discuss the linear stability of the solitary wave whose construction has been just sketched. This stability proof is the key to understanding the Gunn effect in long semiconductors because the field corresponds to that of a solitary wave moving on the steady plateaux (with constant J) for most of each period of the current oscillation [14]. It is easy to find an instability criterion [8]. In a system of coordinates moving with the solitary wave, $\delta = x - Jt$, the electric field is time-independent inside the wave and constant outside the wave. Then a small disturbance can be written as in (3.3) where $\hat{e}$ now depends also on the moving coordinate ξ (see (4.4c) below). If the differential impedance (3.4b) satisfies $Z(0) > 0$, the solitary wave is unstable [8]. A widespread conjecture [8], [32], [1], [3], [9], [19] is that $Z(0) < 0$ implies linear stability of the wave. Incomplete proofs of this conjecture in the literature include Butcher and Rowlands [8], who ignored the possibility of λ being complex, and Volkov and Kogan [32], who used the principle of the argument with an unproved implicit assumption (see [3]; see also [4] for an application of these arguments to the stability of the steady state which clarifies the implicit assumptions). In view of this situation, we have attempted to solve the linearized equation without separating variables, and have obtained only partial results.

Outside the shock and corner layers of the pulse and far from the contacts, a disturbance from the electric field and the current corresponding to a solitary wave moving on $E_1(J)$ satisfies (3.2a)–(3.2c), where E is now

$$E = E_+(J) - \xi, \quad \text{when } 0 < \xi < E_+(J) - E_1(J), \tag{4.4a}$$

$$E = E_1(J), \quad \text{otherwise}. \tag{4.4b}$$

Near the contacts, (4.4b) has to be modified in an obvious way for E to satisfy the boundary conditions. In (4.4a),

$$\xi = x - x_0 - Jt, \tag{4.4c}$$

where $x = x_0$ is the location of the shock at $t = 0$.

We have not been able to find a direct proof of the linear stability of the solitary wave except in a case where the disturbance vanishes outside the pulse and the diffusivity is zero. Consider the following simple curve $v(E)$ instead of (1.4):

$$v(E) = v_0 - \gamma E \quad \text{when } E > 0 \quad (\gamma > 0), \tag{4.5a}$$

$$0 \le v(E) \le v_0 \quad \text{when } E = 0. \tag{4.5b}$$

Assume that the resistivities of the contacts are zero. We operate in a region $E < v_0/\gamma$ so that $v(E) > 0$. Then the field of the moving pulse is given by (4.4a)–(4.4c) (with $E_1(J) = 0$) outside the shock and corner layers, (4.3) is an exact equation and J is given by

$$J = v_0 - \gamma\sqrt{\frac{\phi}{2}}\,. \tag{4.6}$$

Outside the support of the solitary wave, $v'(E) = +\infty$, and the field disturbance is zero. Inside the support of the solitary wave (3.2a) becomes (after a trivial calculation)

$$\frac{\partial \tilde{e}}{\partial t} + (\gamma\xi - v_0 + J)\frac{\partial \tilde{e}}{\partial \xi} = \tilde{j}(t). \tag{4.7}$$

Notice that $\tilde{e} = \tilde{e}(\xi, t)$ because the field profile (4.4) is time-independent inside the wave and constant outside the wave. We shall ignore in what follows the shock and corner layers of the wave. Requiring the continuity of $\tilde{e}$ to be maintained at the point $\xi = E_+$, where the front cuts the baseline, we obtain

$$\tilde{e}(E_+, t) = 0. \tag{4.8}$$

We finally pose an initial disturbance

$$\tilde{e}(\xi, 0) = f(\xi), \quad 0 < \xi < E_+; \quad f(E_+) = 0. \tag{4.9}$$

Equation (4.7) is solved by the method of characteristics

$$\tilde{e}(\xi,t) = f(\xi_0(\xi,t)) + \int_0^t \tilde{j}(s)\,ds, \tag{4.10}$$

where

$$\xi_0(\xi,t) = \frac{v_0 - J}{\gamma}(1 - e^{-\gamma t}) + \xi e^{-\gamma t}. \tag{4.11}$$

As $t \to \infty$, $\xi_0 \to (v_0 - J)/\gamma$ and (4.10) converges rapidly:

$$\tilde{e}(\xi,t) \to f\left(\frac{v_0 - J}{\gamma}\right) + \int_0^t \tilde{j}(s)\,ds. \tag{4.12}$$

The disturbance in the current can be calculated from the condition (3.2c) so that the integral of the field disturbance vanishes:

$$\int_0^{E_+} f(\xi_0(\xi,t))\,d\xi + \int_0^{E_+}\int_0^t \tilde{j}(s)\,d\xi\,ds = 0. \tag{4.13}$$

We change the integration variable to ξ_0 in the first integral

$$e^{\gamma t}\int_{\xi_0(0,t)}^{\xi_0(E_+,t)} f(\xi_0)\,d\xi_0 + E_+\int_0^t \tilde{j}(s)\,ds = 0. \tag{4.14}$$

Letting $t \to \infty$, we observe from (4.11) that both limits in the first integral in (4.14) tend to the value $(v_0 - J)/\gamma$ and

$$\xi_0(E_+,t) - \xi_0(0,t) = E_+ e^{-\gamma t}. \tag{4.15}$$

Inserting these in (4.14) and dividing by E_+ we obtain

$$f\left(\frac{v_0 - J}{\gamma}\right) + \int_0^t \tilde{j}(s)\,ds \to 0 \quad \text{as } t \to \infty. \tag{4.16}$$

Combining (4.12) and (4.16), we obtain that the disturbance $\tilde{e}(\xi,t) \to 0$ exponentially fast as $t \to \infty$. This proves the linear stability of the solitary wave in the idealized case. Extending this proof to the general case involves studying the linearized equation also in the shock and corner layers of the wave [1], and, more importantly, using a general $v(E)$. Further work in this direction is now under way.

5. Discussion. In this paper we have constructed the steady state of the classical drift-diffusion model of Gunn effect in GaAs, asymptotically as $L \to +\infty$, $\delta \downarrow 0$ ($L\delta \downarrow 0$). We have analyzed the linear stability of the steady state with vanishing diffusivity both for finite and for infinite sample length. The diagram ϕ/L versus L of Fig. 3 shows that above a minimal length there is a voltage interval $(\phi_\alpha, \phi_\omega)$ outside which the steady state is linearly stable. The width of this interval grows with L, and its lower end, $\phi_\alpha(L)$, tends rapidly to the value $E_1(J_c)L$ as L increases. These results for infinite L confirm the criterion that for the steady state to become unstable, J has to be very close to J_c, the current at which (for high enough contact resistivity ρ) the velocity $v(E)$, and the contact characteristic curve E/ρ intersect [28].

For large L, our work yields expressions for the critical voltages ϕ_α and ϕ_ω, the frequencies of the eigenmodes that become unstable, and the dispersion relation between these frequencies, $(\phi - \phi_\alpha)$ and the real part of the eigenvalues which provide the growth rate of the instability. Previously most of the results were obtained assuming a piecewise linear velocity curve, and no dispersion relation was available.

The results near the onset of the instability $\phi = \phi_\alpha$ depend on the shape of the velocity $v(E)$ for $E \in [E_1(J_c), E_2(J_c)]$, a region where a N-shaped $v(E)$ (such as (1.4) and a $v(E)$ that saturates for large E are similar. Thus our onset results are also valid for a saturating velocity curve [28], [29]. Where our results are specific of a N-shaped $v(E)$ is near the end of the instability, $\phi = \phi_\omega$. A separate analysis is necessary to give asymptotic estimations of ϕ_ω and the corresponding dispersion relation for saturating velocity curves.

The dispersion relation shows that the eigenvalues (of the problem linearized about the steady state) that acquire a positive real part have a vanishing frequency, cross with positive speed the imaginary axis and form a quasicontinuum as $L \to +\infty$. This suggests that the steady state loses stability via Hopf bifurcation, which will be confirmed in a different paper [5].

We have also considered the linear stability of the solitary wave moving with constant current on the steady state, which is the situation for most of each period of the Gunn oscillations. We have proved stability of the wave for an idealized case of zero diffusion and a piecework linear velocity curve.

Appendix A. Proof of Lemma 1. To simplify the writing of the equation, we shall omit explicit mention of the J-dependance of $E(x; J)$ in this Appendix. To derive (2.10a) we write the solution of (2.1) and (2.2) as

$$x = \rho J - E(x) + J \int_{\rho J}^{E(x)} \frac{ds}{J - v(s)},$$

from which

$$x = \rho J - E(x) - \frac{J}{v_2'} \ln\left(\frac{E_2 - E(x)}{E_2 - \rho J}\right) - \frac{J}{v_1'} \ln\left(\frac{E(x) - E_1}{\rho J - E_1}\right) + J \int_{\rho J}^{E(x)} \left(\frac{v_2'^{-1}}{s - E_2} + \frac{v_1'^{-1}}{s - E_1} - \frac{1}{v(s) - J}\right) ds. \tag{A.1}$$

As $J \sim J_c$, the profile of $E(x)$ will be a plateaux at field $E_2(J)$ (starting at $x = 0$) joined through a transition layer of width $O(1)$ to another plateaux at field $E_1(J)$ that reaches $x = L$ (we ignore the diffusive boundary layer). The integral in (A.1) is finite even when $E(x) \to E_1$ and/or $\rho J \to E_2$. Now let $X \in (0, L)$ be the "center" of the transition layer defined so that $E(X) = E_0$, fixed. Then

$$x - X = E(X) - E(x) - \frac{J}{v_2'} \ln\left(\frac{E_2 - E(x)}{E_2 - E(X)}\right) - \frac{J}{v_1'} \ln\left(\frac{E(x) - E_1}{E(X) - E_1}\right) + J \int_{E(X)}^{E(x)} \left(\frac{v_2'^{-1}}{s - E_2} + \frac{v_1'^{-1}}{s - E_1} - \frac{1}{v(s) - J}\right) ds. \tag{A.2}$$

Equation (A.2) can be used to approximate the stationary solution at the two ends of the transition layer between $E_2(J)$ and $E_1(J)$. As $(x - X) \to -\infty$, we leave the transition layer and $E(x) \sim E_2(J)$. By inserting this value at the second logarithmic term and at the upper limit of the integral in (A.2) and solving the resulting expression for $E(x)$, we find (2.10a), where the current J is close, but not identical, to J_c. (2.10b) is found similarly by noticing that $E(x) \sim E_1(J)$ as $(x - X) \to +\infty$.

Equation (2.9b) follows from (2.10a) where we set $x = 0$, $E(0) = \rho J = \rho J_c + \rho(J -$

J_c), and

$$E_2(J) \sim \rho J_c + E_2'(J_c)(J - J_c) = \rho J_c + \frac{J - J_c}{v_2'(E_2(J_c))}, \tag{A.3a}$$

$$\rho(J - J_c) \sim \frac{J - J_c}{v_2'} - c_L \exp\left\{\frac{v_2' X}{J_c}\right\}. \tag{A.3b}$$

Notice that, to leading order, we can substitute $J = J_c$ in c_L and in the exponential term of (2.10a), (2.10b), and (A.3b).

To derive (2.9a), we decompose the integral of the field in (2.3) as

$$\phi = \int_0^{X-\Delta x} E(x)\,dx + \int_{X-\Delta x}^{X} E(x)\,dx + \int_X^{X+\Delta x} E(x)\,dx + \int_{X+\Delta x}^{L} E(x)\,dx, \tag{A.4}$$

choosing $1 \ll \Delta x \ll X$. By using (2.10a) and (2.10b), we obtain

$$\int_0^{X-\Delta x} E(x)\,dx \sim E_2(X - \Delta x) + \frac{c_L J}{v_2'}\left(\exp\left[\frac{v_2'\Delta x}{J}\right] - \exp\left[\frac{v_2' X}{J}\right]\right), \tag{A.5a}$$

(A.5b)

$$\int_{X+\Delta x}^{L} E(x)\,dx \sim E_1(L - X - \Delta x) + \frac{c_R J}{v_1'}\left(\exp\left[-\frac{v_1'\Delta x}{J}\right] - \exp\left[-\frac{v_1'(L-X)}{J}\right]\right).$$

We now lump the exponentially decreasing terms of (A.5a) and (A.5b) into an $o(1)$ error term and substitute these expressions in (2.3). The result is

(A.6)

$$\phi = E_1 L + (E_2 - E_1)X + \int_{X-\Delta x}^{X} [E(x) - E_2]\,dx + \int_X^{X+\Delta x} [E(x) - E_1]\,dx + o(1).$$

Finally, we change variable from x to E in the integrals of (A.6) by means of (2.1), and use $E(X - \Delta x) \sim E_2(J), E(X + \Delta x) \sim E_1(J)$, as $\Delta x \gg 1$. Inserting (A.3a) for $E_2(J)$ and a similar relation for $E_1(J)$ in the result, we obtain (2.9a). Notice that all the improper integrals in Lemma 1 are convergent.

Part (b) of the Lemma can be proven with similar arguments.

Appendix B. The linear stability of the steady state in the limit $\delta \downarrow 0$. In this Appendix we show that the small diffusivity results in additional eigenvalues λ of the linear stability analysis which are of order $1/\delta$ and generically negative. In fact for the boundary layer to have a large effect on the eigenvalues, we have to assume that $\lambda = O(1/\delta)$, so that the linear stability problem

$$-\delta\frac{\partial^2 \hat{e}}{\partial x^2} + \frac{\partial[v(E)\hat{e}]}{\partial x} + \{\lambda + v'(E)\}\hat{e} = j, \tag{B.1a}$$

$$\int_0^L \hat{e}(x;\lambda)\,dx = 0, \tag{B.1b}$$

$$\hat{e}(0;\lambda) = \hat{e}(L;\lambda) = \frac{\rho j}{1 + \lambda\rho} \tag{B.1c}$$

can be approximated in the boundary layer by

(B.2a) $$-\delta^2\frac{\partial^2\hat{e}_{in}}{\partial x^2}+\delta\frac{\partial[v(E)\hat{e}_{in}]}{\partial x}\sim-(\lambda\delta)\hat{e}_{in},\quad \text{i.e.,}$$

(B.2b) $$-\frac{\partial^2\hat{e}_{in}}{\partial \xi^2}+\frac{\partial[v(E)\hat{e}_{in}]}{\partial \xi}=-(\lambda\delta)\hat{e}_{in},$$

with zero-data Dirichlet boundary condition at $\xi=0$ $(x=L)$ and as $\xi\equiv(x-L)/\delta\downarrow-\infty$ (leaving the boundary layer). In (B.2), $E=E_{in}(\xi)$ is given by (1.16). Outside the boundary layer, we can approximate (B.1a) and (B.1c) by

(B.3) $$\lambda\hat{e}_{out}\sim j,$$

so that j solves (B.1b):

(B.4) $$\frac{jL}{\lambda}\sim-\delta\int_{-\infty}^{0}\hat{e}_{in}(\xi;\lambda)\,d\xi,\qquad \xi\equiv\frac{x-L}{\delta}.$$

From this expression it is clear that $j=O(1/L)$, which is consistent with having neglected it in (B.2).

We now return to (B.2) on the negative half ξ-axis with Dirichlet data. By a standard change of variable we convert (B.2a) in Schrödinger's equation,

(B.5a) $$-\delta^2\frac{\partial^2 f}{\partial x^2}+w(x;\delta)f\sim-(\lambda\delta)f,$$

(B.5b) $$f(x;\lambda)\equiv\hat{e}(x;\lambda)\exp\left\{-\frac{1}{2\delta}\int v(E)\,dx\right\},$$

(B.5c) $$w(x;\delta)\equiv\frac{1}{4}v(E)^2+\frac{1}{2}\delta v'(E)\frac{dE}{dx}.$$

The number of positive λ (i.e., of negative eigenvalues of the Schrödinger's equation (B.5a) with Dirichlet data) can be estimated by the Courant–Weyl integral [26]

(B.6a) $$N\sim\frac{1}{\pi\delta}\int_{w<0}\sqrt{-w(x;\delta)}\,dx\sim\frac{1}{2\pi}\int_{w<0}\frac{\sqrt{2|v'(E)|Q(E)-v(E)^2}}{Q(E)}\,dE,$$

(B.6b) $$Q(E)=Q(E,E_1)=\int_{E_1}^{E}v(s)\,ds,$$

as $\delta\downarrow 0$ (cf. (1.15)). In the numerical calculations that we have performed for typical parameter values, *all* the large eigenvalues of (B.1a)–(B.1d), $\lambda=O(1/\delta)$, turned out to be negative, thereby corresponding to stable modes. For $\lambda=O(1)$, the boundary layer yields an $O(\delta)$ correction to (B.1b) corresponding to a negligible $O(\delta)$-correction to λ. This justifies ignoring the boundary layer in the stability calculations of this paper and also in the literature [28].

Appendix C. Proof of Lemma 2. Setting $x=0$ in (2.10a) we have

(C.1) $$\rho J\sim E_2(J)-c_L\exp\left(\frac{v_2'X}{J}\right).$$

Then a small variation of X, $\delta X\ll 1$, provokes a much smaller variation of J, $\delta J\ll\delta X$ (see below). Subtracting (C.1) from the corresponding relation for $X=X_c+$

δX and $J = J + \delta J$, we find

$$[\rho - E_2'(J)]\delta J \sim -\frac{c_L v_2'}{J}\exp\left(\frac{v_2' X_c}{J}\right)\delta X. \tag{C.2}$$

We now differentiate the identity

$$J = v(E_2(J))$$

with respect to J obtaining

$$E_2(J) = 1/v_2'. \tag{C.3}$$

Inserting (C.3) in (C.2) and using (3.15), we find

$$\delta J \sim \frac{v_1'(E_2 - E_1)}{L}\,\delta X. \tag{C.4}$$

This relation has been derived by ignoring terms of the order of $\exp(v_2' X_c / J)\delta J$ as compared with the retained term of order δX times the exponential. This is justified by the result (C.4). We now subtract (A.6) evaluated at $\phi = \phi_\alpha + \delta\phi$ from (A.6) evaluated at $\phi = \phi_\alpha$, thereby finding

$$\delta\phi \sim E_1' L \delta J + (E_2 - E_1)\delta X \sim 2(E_2 - E_1)\,\delta X, \tag{C.5}$$

by (C.4) and $E_1(J) = 1/v_1'$, which follows from $J = v(E_1(J))$. Equation (C.5) is (3.20a). Repeating step by step the previous derivation for $\phi = \phi_\omega - \delta\phi$ corresponding to $X = X_c + \delta X$, we find (3.20b).

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