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H-theorem for electrostatic or self-gravitating Vlasov–Poisson–Fokker–Planck systems

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Abstract

The H-theorem and the asymptotic behavior for the Vlasov–Poisson–Fokker–Planck system are found. On bounded domains, boundary conditions defined by a scattering kernel are considered. The distribution function evolves to a Maxwellian solving the Poisson–Boltzmann–Emden equation with Dirichlet or Neumann boundary conditions.

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We have proved an H-theorem for gaseous systems described by the Vlasov–Poisson–Fokker–Planck (VPFP) system. Finding an appropriate Lyapunov functional allows us to show that initial conditions evolve to the stationary solutions of the problem. The VPFP system is a kinetic description of a gas of particles whose momenta change only slightly during collision events and which are subject to the self-consistent field created by the particles themselves. Two typical examples follow.

(i) *Electrostatic case*: slow momentum relaxation of a small admixture of heavy positively charged ions in a light gas of neutral particles (which is considered to be in equilibrium). The low density of the heavy particles implies that their collisions with one another may be neglected, whereas their momenta change little when colliding with the light particles of the environment. Under these conditions the collision term in the kinetic equation may be approximated by the Fokker–Planck form [1].

(ii) *Gravitational case*: a self-gravitating system undergoing diffusion in velocity space due to the fluctuations between the actual force acting on each particle and the mean-field force given by the Poisson equation (soft collisions) [2].

These cases are described by the system

$$\frac{\partial f}{\partial t} + (v\cdot\nabla_x)f - \nabla_v\cdot[(\nabla_x\Phi+\beta v)f] - \sigma\nabla_v^2 f = 0, \tag{1}$$

$$-\nabla_x^2\Phi(t,x) = \theta\rho(f)(t,x). \tag{2}$$

Here $f(t,x,v)\geqslant 0$ is the distribution function, and $\Phi(t,x)$ is the internal (mean-field) potential (electrostatic if $\theta=1$ or gravitational if $\theta=-1$; $\rho(f)(t,x)=\int f(t,x,v)\,dv$ is the charge or mass density). We do

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not consider the effect of an external potential on our system for the sake of simplicity although our results could be extended also to that situation. The term $\mathrm{div}_v(-\beta v f)$ represents dynamical friction [2], and the friction and diffusion coefficients obey the Einstein relation: $\sigma/\beta = k_B T/m$, where k_B is the Boltzmann constant, T the temperature of the thermal bath and m the mass of the particles.

The system either extends to infinity (and then f and Φ decay to zero) or it is inside some enclosure Ω with general boundary conditions for f on its boundary $\partial\Omega$,

$$f(t,x,v) = \int_{\Gamma_+^x} R(t,x;v,v^*) f(t,x,v^*)\,\mathrm{d}v^*. \tag{3}$$

We consider that Φ is constant on the boundary $\partial\Omega$: for a connected boundary this condition corresponds to a perfect conductor in the electrostatic case. The case of disconnected boundaries with different constant values of Φ is also included. Most of our results also hold if Φ is constant on parts of $\partial\Omega$ and the normal component of the internal field vanishes on the rest (Dirichlet–Neumann boundary condition). (3) is an integral relation between the distribution of particles coming out of an infinitesimal section of the boundary at a given time, with velocity v, and the density of the particles impinging upon the same boundary section. (We use the notation $\Gamma_\pm^x = \{v \in \mathbb{R}^3$ such that $\mathrm{sgn}(v \cdot n(x)) = \pm 1, x \in \partial\Omega\}$ and $\Gamma_\pm = \{(x,v) \in \partial\Omega \times \mathbb{R}^3$ such that $\mathrm{sgn}(v \cdot n(x)) = \pm 1\}$, where $n(x)$ is the unit normal outward on the boundary of the domain Ω at $x \in \partial\Omega$). The scattering kernel $R(t,x;v,v^*)$ has the following properties [3]:

(i) R is always positive.

(ii) R satisfies the following normalization condition for any $v^* \in \Gamma_+^x$,

$$|v^* \cdot n(x)| = \int_{\Gamma_-^x} R(t,x;v,v^*) |v \cdot n(x)|\,\mathrm{d}v. \tag{4}$$

(iii) As a consequence of the reciprocity principle (see Ref. [3]), the relation

$$M(v) = \int_{\Gamma_+^x} R(t,x;v,v^*) M(v^*)\,\mathrm{d}v^* \tag{5}$$

holds for any $v \in \Gamma_-^x$. $M(v) = (2\pi\sigma/\beta)^{-3/2} \times \exp(-\beta|v|^2/2\sigma)$ is the Maxwellian at the wall with the temperature σ/β of the thermal bath surrounding the particles.

Notice that the classical cases of specular and reverse reflection ("bounce-back" boundary condition [4]) are included in this definition. There are several works devoted to the construction of more realistic scattering kernels which satisfy the above conditions (see Section 8.4 of Ref. [3] and references therein). The physical relevance of properties (i)–(iii) is also discussed there.

By integrating the VPFP system and using these boundary conditions, one can prove a number of balance laws for the energy, entropy and mass. The later is the continuity equation

$$\frac{\partial\rho(f)}{\partial t} + \nabla_x \cdot j(f) = 0, \quad j(f) = \int_{\mathbb{R}^3} v f\,\mathrm{d}v. \tag{6}$$

Also, the boundary conditions on f yield interesting properties for the moments of f on $\partial\Omega$. For instance, condition (ii) implies that $j(f) \cdot n$ is zero on the boundary $\partial\Omega$. Notice that this result is not physically realistic in the electrostatic case: if the boundary is a perfect conductor and our system is out of equilibrium, we would expect a nonzero normal component of the current density on the boundary corresponding to particles leaving or entering Ω through the contacts. To discuss this case we would need more general boundary conditions for f and it is unclear to us whether an H-theorem could then be proved.

It is well known that the relative entropy between f and the stationary solutions of the Fokker–Planck equation is the appropriate Lyapunov functional used to derive the H-theorem [5,6]. In contrast to the usual Fokker–Planck equation, which is linear in f, the VPFP system is nonlinear and the relative entropy is not a Lyapunov functional. Examples of nonlinear Fokker–Planck equations with known Lyapunov functionals include models of synchronization of oscillator populations with mean-field coupling [7–9]. These examples have drift terms that depend linearly on a moment of the distribution function, which is also the case for the VPFP system. Thus, we derive a Lyapunov functional for the VPFP system using similar ideas [7]:

(i) Define a relative entropy with respect to a non-normalized "stationary" distribution

$$\eta(t) = \int_{\Omega}\int_{\mathbb{R}^3} f \log(f/\tilde{f})\, dx\, dv, \qquad (7)$$

where $\tilde{f}$ satisfies the equation $(v\cdot\nabla_x)\tilde{f} - \nabla_v\cdot[(\nabla_x\Phi + \beta v)\tilde{f}] - \sigma\nabla_v^2\tilde{f} = 0$, with the internal potential Φ given by solving (2) with the exact distribution f. $\tilde{f}$ is given by

$$\tilde{f}(t,x,v) = \exp\left[-\frac{\beta}{\sigma}\left(\frac{|v|^2}{2} + \Phi(t,x) - \frac{1}{M}\mu(t)\right)\right], \qquad (8)$$

where $M = \int f\, dx\, dv$ is the total mass of the system.

(ii) Find $\mu(t)$ so that $\int f(\partial \log\tilde{f}/\partial t)\, dx\, dv = 0$.

(iii) Show that $\eta'(t) \leqslant 0$.

(iv) Show that $\eta(t)$ is bounded below.

After integrating by parts, using the Poisson and continuity equations and taking into account that the surface integral $\int_{\partial\Omega}(\partial\Phi/\partial t) n(x)\cdot\nabla_x\Phi\, dS(x)$ is zero, Step (ii) of the previous procedure yields (up to an additive constant)

$$\mu(t) = \tfrac{1}{2}\theta \int_{\Omega} |\nabla_x\Phi(t,x)|^2\, dx. \qquad (9)$$

By inserting Eqs. (8) and (9) in (7), we see that $\eta(t)$ is the free-energy functional,

$$\eta(t) = \int_{\Omega}\int_{\mathbb{R}^3} f\log f\, dx\, dv + \frac{\beta}{2\sigma}\int_{\Omega}\int_{\mathbb{R}^3} |v|^2 f\, dx\, dv + \frac{\beta\theta}{2\sigma}\int_{\Omega} |\nabla_x\Phi(t,x)|^2\, dx. \qquad (10)$$

We now check (iii). By taking the derivative of $\eta(t)$, we find $\eta'(t) = \int(\partial f/\partial t)\log(f/\tilde{f})\, dx\, dv$, plus two terms that are zero due to mass conservation and Step (ii) in the definition of $\eta(t)$. We now insert the VPFP system (1) and (2), written as

$$\frac{\partial f}{\partial t} = \sigma\,\mathrm{div}_v\left(\nabla_v f - f\nabla_v\log(\tilde{f}) - \frac{f}{\beta}\nabla_x\log(\tilde{f})\right) - v\cdot\nabla_x f$$
$$= \sigma\,\mathrm{div}_v\left[\left(\nabla_v\log(f/\tilde{f}) - \frac{1}{\beta}\nabla_x\log(\tilde{f})\right) f\right] - v\cdot\nabla_x f,$$

into the expression for $\eta'(t)$ and integrate by parts (retaining the boundary terms on $\partial\Omega$). The result is

$$\eta'(t) = -\sigma\int_{\Omega}\int_{\mathbb{R}^3} f|\nabla_v\log(f/\tilde{f})|^2\, dx\, dv - \int_{\partial\Omega}\int_{\mathbb{R}^3}(v\cdot n) f(\log f - \log\tilde{f})\, dS\, dv + \frac{\beta}{\sigma}\int_{\Omega}\int_{\mathbb{R}^3} f(\nabla_x\log\tilde{f}\cdot\nabla_v\log f - \nabla_x\log f\cdot\nabla_v\log\tilde{f})\, dx\, dv.$$

We now use Eq. (8), the divergence theorem and the definition of the current density $j(f)$ to obtain

$$\eta'(t) = -\sigma\int_{\Omega}\int_{\mathbb{R}^3} f|\nabla_v\log(f/\tilde{f})|^2\, dx\, dv - \int_{\partial\Omega} n\cdot j(f)\left(1 + \frac{\beta\mu}{\sigma M} - \frac{\beta\Phi}{\sigma}\right) dS - \int_{\partial\Omega}\int_{\mathbb{R}^3} v\cdot n\left(\frac{\beta}{2\sigma}|v|^2 + \log f\right) f\, dS\, dv. \qquad (11)$$

(See Ch. 6 of Ref. [6] for similar manipulations.) The first term in (11) is nonnegative, while the second term is zero because $j(f)\cdot n = 0$ on $\partial\Omega$. By using Jensen's inequality one can prove that the third term in (11) is also nonnegative (see p. 241 of Ref. [3]; notice that the authors denote by $n(x)$ the *inward* normal to $\partial\Omega$, contrary to our notation). Of course this term is zero for the case of infinite space with natural boundary conditions. Thus we have shown that $\eta'(t) \leqslant 0$. That $\eta(t)$ is bounded below follows from the inequality $\alpha\log(\alpha) \geqslant \alpha - 1$ (for any $\alpha > 0$) and conservation of mass if $\exp[-\beta\Phi(t,x)/\sigma]$ is integrable. In this point, the electrostatic and gravitational cases are different:

– *Electrostatic case.* By using the maximum principle on the Poisson equation (2) with a positive right-hand side, it is straightforward to show that Φ is always positive. Thus, the integrability of $\exp[-\beta\Phi(t,x)/\sigma]$ in bounded domains holds.

– *Gravitational case.* In order to show the boundedness below the free energy, we must assume that the potential energy of the system, i.e.,

$$\int_\Omega |\nabla_x \Phi(t,x)|^2 \,\mathrm{d}x,$$

remains bounded in time.

These results are proved rigorously for bounded Ω in Ref. [10].

We now prove that the distribution function tends to stationary states as $t \to \infty$. Given that $\eta(t)$ is a Lyapunov functional, time sequences of f will tend to a function $f_\infty(t,x,v)$ such that $\eta'(t) = 0$. (11) then implies that $f_\infty/\tilde{f}$ cannot depend on v. Hence, $f_\infty(t,x,v) = \exp(-\beta|v|^2/2\sigma) g_\infty(t,x)$. Inserting this expression in the VPFP system we have

$$\frac{\partial}{\partial t} g_\infty = -v \cdot \left(\nabla_x g_\infty + \frac{\beta}{\sigma} g_\infty \nabla_x \Phi_\infty \right),$$

where Φ_∞ is the solution of the Poisson equation (2) associated to $\rho(f_\infty)$. As $g_\infty(t,x)$ does not depend on v, the right-hand side is a linear function of v and the left-hand side is independent of v; this is contradictory, unless both sides vanish. Hence we have $\partial g_\infty/\partial t = 0$ and $\nabla_x g_\infty + (\beta/\sigma) g_\infty \nabla_x \Phi_\infty = 0$, which yields the following stationary solution of the VPFP system with mass M,

$$f_\infty(x,v) = \left(\frac{2\pi\sigma}{\beta}\right)^{-3/2} \times M \frac{\exp\{-(\beta/\sigma)[\frac{1}{2}|v|^2 + \Phi_\infty(x)]\}}{\int_\Omega \exp[-(\beta/\sigma)\Phi_\infty(x')]\,\mathrm{d}x'}.$$

Here $\Phi_\infty(x)$ solves the Boltzmann–Poisson–Emden problem [11]

$$-\nabla_x^2 \Phi_\infty = \theta M \frac{\exp[-(\beta/\sigma)\Phi_\infty]}{\int_\Omega \exp[-(\beta/\sigma)\Phi_\infty]\,\mathrm{d}x}, \qquad (12)$$

with boundary conditions $\Phi_\infty = 0$ on $\partial\Omega$ (or Neumann boundary conditions).

The existence properties of the stationary problem (12) with Dirichlet boundary conditions depend strongly on the interaction type.

(i) *Electrostatic case.* In this case the corresponding Euler–Lagrange functional is strictly convex (see p. 140 of Ref. [11]). As a consequence, we have a unique possible steady state. We conclude that the asymptotic behavior of the VPFP system is given by the unique stationary limit.

(ii) *Gravitational case.* Using Eq. (12) the function $u = -(\beta/\sigma)\Phi$ satisfies

$$-\Delta u = \alpha \frac{\mathrm{e}^u}{\int_\Omega \mathrm{e}^u \,\mathrm{d}x}, \qquad (13)$$

with Dirichlet boundary conditions, where $\alpha = (\beta/\sigma)M$. Whether Eq. (13) has solution(s) depends strongly on the topology of the domain Ω (see Refs. [12,13]). In fact, if Ω is the unit ball $B(0,1)$, we have the following properties:

- There exists a value of the parameter $\alpha_\star$ such that Eq. (13) with Dirichlet boundary conditions has at least one solution for any $0 \leqslant \alpha < \alpha_\star$ and no solution for $\alpha > \alpha_\star$. In the latter case the potential energy of the system is not bounded, i.e,

$$\lim_{t\to\infty} \int_\Omega |\nabla_x \Phi(t,x)|^2 \,\mathrm{d}x = \infty.$$

- For $\alpha = \alpha_0 = 2\,\mathrm{meas}[\partial B(0,1)]$, Eq. (13) has infinitely many bounded solutions (starting with a minimal one), and a unique unbounded radial solution $u = U(x) = -2\log|x|$.
- For α small enough, the solution is unique.

On the other hand, if Ω is an annulus, Eq. (13) has at least a solution for any value of the parameter α (see Ref. [12]).

When α is such that (13) has more than one solution, it is an open problem to determine the dynamical behavior of the distribution function $f(t,x,v)$ as t approaches infinity. For instance, we do not know whether it is possible for two different time sequences of the distribution function to approach different stationary states.

The results reported in this paper can be made mathematically rigorous as we will explain elsewhere [10].

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