# High-field limit of the Vlasov-Poisson-Fokker-Planck system: A comparison of different perturbation methods

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A reduced drift-diffusion (Smoluchowski-Poisson) equation is found for the electric charge in the high-field limit of the Vlasov-Poisson-Fokker-Planck system, both in one and three dimensions. The corresponding electric field satisfies a Burgers equation. Three methods are compared in the one-dimensional case: Hilbert expansion, Chapman-Enskog procedure and closure of the hierarchy of equations for the moments of the probability density. Of these methods, only the Chapman-Enskog method is able to systematically yield reduced equations containing terms of different order.



## I. INTRODUCTION

The Vlasov-Poisson-Fokker-Planck (VPFP) system describes the probability density $\rho$ of an ensemble of electrically charged Brownian particles in contact with a thermal bath and subject to their self-consistent electric field $\phi$, see [1]. It consists of the equations:

$$\frac{\partial\rho}{\partial t} + v\cdot\nabla_x\rho - \nabla_v\cdot\left[\left(\mu v + \frac{e}{m}\nabla_x\phi\right)\rho + \frac{\mu k_BT}{m}\nabla_v\rho\right] = 0\,, \tag{1}$$

$$\varepsilon\,\Delta_x\phi = -eN_0\int\rho\,dv, \tag{2}$$

$$\int\rho(x,v,t)\,dx\,dv = 1. \tag{3}$$

Here $N_0$ is the number of particles, $\mu$ a friction coefficient, $e$ and $m$ particle charge and mass, $T$ temperature and $\varepsilon$ permittivity. Sometimes it is interesting to consider the high-field limit, in which the collision term (divergence with respect to $v$) in (1) dominates the other two. In this case, $\rho$ quickly evolves toward a displaced Maxwellian,

$$\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\exp\left[-\frac{m\left(v+\frac{e}{m\mu}\nabla_x\phi\right)^2}{2k_BT}\right]$$

times a slowly-varying function, $P(x,t)$. Essentially, $P$ is a charge density whose evolution we would like to describe. Appropriate scales in which this fast equilibration to the displaced Maxwellian occurs may be found as follows. A typical velocity is found by balancing friction, $\mu v\rho$ and flux $\mu k_BT\nabla_v\rho/m$, which yields $\mu v_0 = \mu k_BT/(mv_0)$, or

$$v_0 = \sqrt{\frac{k_BT}{m}}\,, \tag{4}$$

which is the thermal velocity. If we impose that $\mu v$ and $e\nabla_x\phi/m$ be of the same order, we find the unit of electric field, $E_0$,

$$E_0 = \frac{\mu m v_0}{e} = \frac{\mu v_0}{e}\sqrt{mk_BT}\,. \tag{5}$$

The Poisson equation yields the unit of length, $\varepsilon E_0/x_0 = eN_0/x_0^3$, or

$$x_0 = \sqrt{\frac{eN_0}{\varepsilon E_0}} = e\left(\frac{N_0}{\mu\varepsilon}\right)^{\frac{1}{2}}(mK_BT)^{-\frac{1}{4}}\,. \tag{6}$$

The relation between the second and third terms in (1) yields the small parameter

$$\epsilon = \frac{v_0}{\mu x_0} = \frac{1}{e}\left(\frac{\varepsilon^2(k_BT)^3}{m(\mu N_0)^2}\right)^{\frac{1}{4}}\,. \tag{7}$$

The unit of $\rho$ should be $(x_0v_0)^{-3}$ because of the normalization condition (3). Finally, the unit of time should be $x_0/v_0$ provided we want the first two terms in (1) to be of the same order. Nondimensionalizing the system (1) - (3) in these units, we can rewrite it as

$$\nabla_v\cdot\left[(v+\nabla_x\phi)\,\rho + \nabla_v\rho\right] = \epsilon\left(\frac{\partial\rho}{\partial t} + v\cdot\nabla_x\rho\right)\,, \tag{8}$$

$$\Delta_x\phi = -\int\rho\,dv, \tag{9}$$

$$\int\rho\,dx\,dv = 1, \tag{10}$$

to be solved together with appropriate initial conditions and decay boundary conditions at infinity.

We are interested in finding a simpler reduced equation for $P(x,t)$ in the high-field limit as $\epsilon\to 0+$. This problem has been recently tackled by Nieto, Poupaud and Soler, [2], who proved rigorously that $P$ obeys the following hyperbolic system (in dimension one):

$$P_t = \frac{\partial}{\partial x}(\phi_x P), \quad (11)$$

$$\phi_{xx} = -P, \quad (12)$$

$$\int P\, dx = 1, \quad (13)$$

to leading order as $\epsilon \to 0+$ (subscripts mean partial derivation with respect to the indicated variable). By using the decay of $P$ at infinity, they derived the following equation for the electric field, $E = -\phi_x$:

$$E_t + E\, E_x = 0. \quad (14)$$

As is well-known, this equation may develop shock waves in finite time. Nieto *et al* proved that there exists a unique entropy solution of the VPFP system in the high-field limit, and that the shock velocity corresponds to interpreting (14) as the conservation law $E_t + \frac{1}{2}(E^2)_x = 0$. The uniqueness result suggests that this conservation law could be regularized by adding a small viscosity term, $\epsilon\, E_{xx}$. Then analysis of the resulting Burgers equation in the limit as $\epsilon \to 0+$ yields the unique entropy solution. Their method amounted to closing the hierarchy of equations for the moments of $\rho$, and leaves open the question of how to add higher-order terms to the system (11) - (13), or (which is related) how to regularize (14).

In this paper, we shall consider these problems. To this end, we shall compare three classical approaches in kinetic theory, the Hilbert expansion, closure of moment equations and the Chapman-Enskog method (CEM) as applied to the one-dimensional VPFP system. These approaches yield the same result for the parabolic scaling

$$\nabla_v \cdot (v\,\rho + \nabla_v \rho) = \epsilon\,(v \cdot \nabla_x \rho - \nabla_x \phi \cdot \nabla_v \rho) + \epsilon^2\, \frac{\partial \rho}{\partial t}, \quad (15)$$

which is interesting and usually employed in kinetic theory [3,4]. This scaling corresponds to lower values of the electric field, so that the diffusive and frictional terms in the VPFP equation are in fact much larger than the corresponding term in (1). For the present model, the parabolic scaling was also studied rigorously by Poupaud and Soler [5], who derived from (15) a drift-diffusion for $P = \int \rho\, dv$ in the limit as $\epsilon \to 0+$. In the high-field limit corresponding to the scaling (8), we shall see that only the CEM yields satisfactory results *systematically.* In fact, the Hilbert expansion yields the following additional system for the order $\epsilon$ correction to $P$, $P^{(1)}$:

$$P^{(1)}_t - \frac{\partial}{\partial x}(\phi^{(0)}_x P^{(1)} + \phi^{(1)}_x P) = P_{xx}, \quad (16)$$

$$\phi^{(1)}_{xx} = -P^{(1)}, \quad (17)$$

$$\int P^{(1)}\, dx = 0. \quad (18)$$

At face value, this equation breaks down in the shock regions. However, it is compatible with the result of the CEM,

$$P_t - \frac{\partial}{\partial x}(\phi_x P) = \epsilon\, P_{xx}, \quad (19)$$

$$\phi_{xx} = -P, \quad (20)$$

$$\int P\, dx = 1, \quad (21)$$

provided we substitute $P$ by $P + \epsilon P^{(1)}$ and equate like powers of $\epsilon$ in both sides of these equations. This situation is exactly that found when using the method of multiple scales to describe certain codimension two bifurcations [6,7]. The key point is that the CEM yields reduced equations which may contain terms of different order in $\epsilon$, unlike the other methods. Appropriate closure of the equations for the moments of $\rho$ may yield (19) - (21), but it is not a systematic method.

The rest of the paper is as follows. The Hilbert expansion and the CEM are applied to the one-dimensional VPFP system in Sections II and III, respectively. The latter section also contains the result of applying the Chapman-Enskog procedure to the three-dimensional problem. Section IV shows how to obtain the same results by closing the hierarchy of equations for the moments. Finally Section V contains our conclusions.

## II. THE HILBERT EXPANSION

It consists of inserting the power series

$$\rho = \rho^{(0)} + \epsilon \rho^{(1)} + \epsilon^2 \rho^{(2)} + O(\epsilon^3), \quad (22)$$

in the VPFP system. This expansion is really akin to the method of multiple scales with slow time scale $t$ [8]. We assume that the leading order contribution to the solution has already relaxed in the fast time scale. Equating like powers of $\epsilon$ in both sides of the one-dimensional VPFP system, we obtain

$$\frac{\partial}{\partial v}\left\{\frac{\partial \rho^{(0)}}{\partial v} + [v + \phi^{(0)}_x]\,\rho^{(0)}\right\} = 0, \quad (23)$$

$$\frac{\partial^2 \phi^{(0)}}{\partial x^2} = -\int_{-\infty}^{\infty} \rho^{(0)}\, dv, \quad (24)$$

$$\int_{-\infty}^{\infty} \rho^{(0)} dx dv = 1, \quad (25)$$

$$\mathcal{L}\rho^{(1)} \equiv \frac{\partial}{\partial v}\left\{\rho^{(1)}_v + [v + \phi^{(0)}_x]\,\rho^{(1)} + \phi^{(1)}_x]\,\rho^{(0)}\right\} = \rho^{(0)}_t + v\rho^{(0)}_x, \quad (26)$$

$$\phi^{(1)}_{xx} = -\int_{-\infty}^{\infty} \rho^{(1)}\, dv, \quad (27)$$

$$\int_{-\infty}^{\infty} \rho^{(1)} dx dv = 0, \quad (28)$$

$$\mathcal{L}\rho^{(2)} = \rho_t^{(1)} + v\rho_x^{(1)} - \phi_x^{(1)}\rho_v^{(1)}, \tag{29}$$

$$\phi_{xx}^{(2)} = -\int_{-\infty}^{\infty} \rho^{(2)}\, dv, \tag{30}$$

$$\int_{-\infty}^{\infty} \rho^{(2)} dx dv = 0, \tag{31}$$

and so on.

The solution of (23) - (25) is

$$\rho^{(0)} = \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}}\, P(x,t), \tag{32}$$

for a function $P$ to be determined and such that

$$V = v + \phi_x^{(0)}, \tag{33}$$

$$\phi_{xx}^{(0)} = -P, \tag{34}$$

$$\int_{-\infty}^{\infty} P(x,t)\, dx = 1. \tag{35}$$

Notice that the Fokker-Planck collision term behaves more nicely than the general Boltzmann term. In the high-field limit, the latter may give rise to runaway solutions for many forms of the collision frequency [9].

Since $\int \mathcal{L}\rho^{(n)} dv = 0$, (26) yields the following solvability condition

$$\frac{\partial}{\partial t}\int \rho^{(0)} dv + \frac{\partial}{\partial x}\int v\rho^{(0)} dv = 0.$$

necessary for $\rho^{(1)}$ to be bounded even as $v \to \pm\infty$. This provides

$$P_t = \frac{\partial}{\partial x}\, (\phi_x^{(0)} P), \tag{36}$$

$$\phi_{xx}^{(0)} = -P, \tag{37}$$

together with (13), which coincides with Nieto *et al*'s result, [2].

Suppose we now want to correct the leading order result by going one step further, to solving (26) - (28). The result is

$$\rho^{(1)} = \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}} \left( P^{(1)} - [P_x + \phi_x^{(1)} P]\, V - \frac{V^2-1}{2}\, P^2 \right), \tag{38}$$

$$\phi_{xx}^{(1)} = -P^{(1)}, \tag{39}$$

$$\int P^{(1)} dx = 0, \tag{40}$$

where $P^{(1)}(x,t)$ is yet to be determined. By using the solvability condition for (29), we obtain

$$P_t^{(1)} - \frac{\partial}{\partial x}\, (\phi_x^{(0)} P^{(1)} + \phi_x^{(1)} P) = P_{xx}, \tag{41}$$

to be solved together with (39) and (40). Notice that the left hand side of (41) is a linearization of (36) about $P$. This equation does not make sense at those points where $P$ is discontinuous, but we can easily see that (36) and (41) are obtained by taking $\mathcal{P} \sim P + \epsilon P^{(1)}$ in the following equation:

$$\mathcal{P}_t - \frac{\partial}{\partial x}\, (\phi_x\, \mathcal{P} + \epsilon \mathcal{P}_x) = O(\epsilon^2)\,, \tag{42}$$

to be solved together with the Poisson equation $\phi_{xx} = -\mathcal{P}$ and $\int \mathcal{P} dx = 1$. This situation is exactly that found when using the method of multiple scales to find the normal form describing certain codimension two bifurcations [6,7]. The Hilbert expansion is akin to the method of multiple scales in that it yields reduced equations all whose terms are of the same order. Thus if we insist in calculating higher-order equations, we obtain coupled systems of equations such as those written above. To find directly a reduced equation including terms of different order in $\epsilon$, we should use the CEM. Appropriate closure of the equations for the moments of $\rho$ may yield (42), but not in a systematic manner.

## III. CHAPMAN-ENSKOG METHOD

In one dimension, the dimensionless VPFP system is

$$\frac{\partial}{\partial v}\left\{ \frac{\partial \rho}{\partial v} + [v + \phi_x]\, \rho \right\} = \epsilon\, \left( \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} \right)\,, \tag{43}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\int_{-\infty}^{\infty} \rho\, dv, \tag{44}$$

to be solved together with

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x,v,t)\, dx\, dv = 1, \tag{45}$$

and appropriate initial and decay conditions as $v$ and $x$ tend to $\pm\infty$.

### A. Chapman-Enskog method

Setting $\epsilon = 0$ in (43), we find a simple equation to be solved together with (44) and (45). Its solution is a displaced Maxwellian:

$$\rho = \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}}\, P(x,t), \tag{46}$$

$$V = v + \phi_x^{(0)}, \tag{47}$$

$$\phi_{xx}^{(0)} = -P, \tag{48}$$

$$\int_{-\infty}^{\infty} P(x,t)\, dx = 1. \tag{49}$$

Notice that $P(x,t)$ is an arbitrary function of $x$ and $t$ except for (49). Furthermore, (46) correspond to a particular choice of initial conditions. The Chapman-Enskog ansatz consists of assuming that $\rho$ has the following asymptotic expansion

$$\rho = \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}}\, P(x,t;\epsilon) + \sum_{n=1}^{\infty} \epsilon^n \,\rho^{(n)}(x,v;P). \tag{50}$$

Furthermore, we impose that the amplitude $P$ obeys an equation:

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} \epsilon^n \, F^{(n)}(P), \tag{51}$$

where $F^{(n)}$ are functionals of $P$ to be determined as the procedure goes on. This equation for $P$ is not explicitly written in the usual presentations of CEM [3,4]. Instead, the form of this equation is guessed by writing equations for the moments of $\rho$ and using gradient expansions. We find this latter procedure more confusing.

Insertion of (50) and (51) into the equations and auxiliary conditions yields a hierarchy of linear equations for the $\rho^{(n)}$. Notice that the latter depend on time only through their dependence on $P$. The functionals $F^{(n)}(P)$ are determined so that each equation (and set of auxiliary conditions) for $\rho^{(n)}$ has a solution which is bounded for all values of $v$, even as $v \to \pm\infty$. Once a sufficient number of $F^{(n)}$ is determined, (51) *is the sought amplitude equation*. Please notice that, unlike results from the method of multiple scales, terms in (51) may be of different order.

Let us illustrate how the procedure works by finding $F^{(0)}$ and $F^{(1)}$. Insertion of (50) and (51) in (43), (44) and (10) yields the following hierarchy of linear equations:

$$\begin{aligned}\mathcal{L}\rho^{(1)} &= \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}} \left[ vP_x - vV\phi^{(0)}_{xx}P + F^{(0)} \right. \\ &\quad \left. - \, VP\phi^{(0)}_{xt}\big|_{P_t=F^{(0)}} \right], \end{aligned} \tag{52}$$

$$\begin{aligned}\mathcal{L}\rho^{(2)} &= \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}} \left[ F^{(1)} - VP\phi^{(0)}_{xt}\big|_{P_t=F^{(1)}} \right] + \rho^{(1)}_t \\ &\quad + v\rho^{(1)}_x - \phi^{(1)}_x \,\rho^{(1)}_v \,, \end{aligned} \tag{53}$$

and so on. Before equating like powers of $\epsilon$, we have substituted $P_t = \sum_{n=0}^{\infty} F^{(n)}$ when performing time differentiations such as $\phi^{(0)}_{xt}$. This yields the obvious terms in the hierarchy of equations, which results in rather cluttered formulas, as we ascend in the hierarchy of equations. These equations are to be supplemented by the normalization conditions (49),

$$\int_{-\infty}^{\infty} \rho^{(n)} dv = 0, \;\; n \geq 1, \tag{54}$$

and the linear equations and definitions:

$$\phi^{(n)}_{xx} = -\int_{-\infty}^{\infty} \rho^{(n)} dv\,, \tag{55}$$

$$\mathcal{L}\rho^{(n)} = \frac{\partial}{\partial v}\left[ V\rho^{(n)} + \rho^{(n)}_v + P\,\phi^{(n)}_x \,\frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}}\right], \tag{56}$$

for $n = 1, 2, \ldots$. $V$ is again given by (33).

Let us now consider (52). Since the $v$ integral of its left hand side is zero, this equation has a solution only if the $v$ integral of its right hand side is zero. The corresponding integrals are simplified by using the symmetry of the Maxwellian and shifting integration variables from $v$ to $V$. The condition that the integral of the right side vanish yields

$$F^{(0)} = \left\{ \phi^{(0)}_x \, P \right\}_x . \tag{57}$$

Notice that we needed $F^{(0)}$ in the right side of (52) for this equation to have an appropriate solution.

We now calculate $\phi^{(0)}_{xt}$ in order to simplify the right side of (52). As we explained above, at this order we should substitute $P_t = F^{(0)} = \left\{ \phi^{(0)}_x \, P \right\}_x$ when needed. Hence

$$\begin{aligned}\phi^{(0)}_{xt} &= -K_x * P_t = -K_x * \left\{ \phi^{(0)}_x \, P \right\}_x \\ &= -K_{xx} * \left\{ \phi^{(0)}_x \, P \right\} = -\phi^{(0)}_x \, P, \end{aligned} \tag{58}$$

where $K$ is the Green's function of the one-dimensional Laplacian and $*$ means convolution product. In the previous calculation, we have integrated by parts and used the decay properties of $P$ and the symmetry of $K$. Inserting now (57) and (58) in Eq. (52), we realize that the right hand side thereof is the partial derivative of $(V\phi^{(0)}_{xx}P - P_x)\, e^{-V^2/2}/\sqrt{2\pi}$ with respect to $v$. This immediately yields

$$\rho^{(1)} = -\frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}} \left( \frac{V^2-1}{2}\, P^2 + VP_x \right), \tag{59}$$

which satisfies (54) and yields $\phi^{(1)} = 0$. In (59), we have omitted adding a term like that proportional to $P^{(1)}$ in the right hand side of (38) (satisfying $\int P^{(1)}\, dx = 0$) because all such terms are already included in the amplitude $P(x,t;\epsilon)$.

To find $F^{(1)}$, we insert (59) in (53) and use the solvability condition for this equation. Simplifications arise from the identities

$$\int_{-\infty}^{\infty} \rho^{(1)}_t dv = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho^{(1)} dv = 0,$$

and

$$\int_{-\infty}^{\infty} v\rho^{(1)}_x dv = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} v\rho^{(1)}\, dv.$$

The result is

$$F^{(1)} = P_{xx}\,. \tag{60}$$

We can now insert (57) and (60) into (51) to obtain the sought Smoluchowski equation for $P$:

$$P_t - \frac{\partial}{\partial x}\,(\phi_x\, P + \epsilon P_x) = 0\,, \tag{61}$$

to be solved together with (48). Substituting (48) into (61), and integrating with respect to $x$, we find the following Burgers equation for $E = -\phi_x$:

$$E_t + E\, E_x = \epsilon\, E_{xx}\,. \tag{62}$$

Provided that the initial electric field be uniformly bounded, this Burgers equation has solutions which converge to the distributional solutions of the Hopf equation (14). Thus the CEM is in fact a vanishing-viscosity method which yields a unique solution approaching the unique entropy solution of (14) as $\epsilon \to 0+$. In Ref. [2], the authors proved convergence towards the unique entropy solution of the Hopf equation by means of a different method which used that the limiting electric field is a monotone decreasing function.

### B. Three-dimensional VPFP system

The CEM consists of inserting the following ansatz in (8)

$$\rho = \frac{e^{-\frac{V^2}{2}}}{(2\pi)^{\frac{3}{2}}}\, P(x,t;\epsilon) + \sum_{n=1}^{\infty} \epsilon^n\, \rho^{(n)}(x,v;P). \tag{63}$$

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} \epsilon^n\, F^{(n)}(P), \tag{64}$$

$$V = v + \nabla_x \phi. \tag{65}$$

Then $\phi$ obeys the Poisson equation

$$\Delta_x \phi = -P, \tag{66}$$

and the normalization conditions are

$$\int P(x,t)\, dx = 1, \tag{67}$$

$$\int \rho^{(n)}(x,v;P)\, dx dv = 0. \tag{68}$$

Equations (52) and (53) are obviously generalized to the three-dimensional case. The solvability condition for the first equation yields

$$F^{(0)} = \nabla_x \cdot (\nabla_x \phi\, P).$$

Substituting $F^{(0)}$ in (52) we find

$$\mathcal{L}\rho^{(1)} = \nabla_v \cdot \left\{ \frac{e^{-\frac{V^2}{2}}}{(2\pi)^{\frac{3}{2}}} \left[ P\, v.\nabla_x \nabla_x \phi^{(0)} - \nabla_x P + P \nabla_x \phi_t^{(0)} \right] \right\}.$$

The solution is then

$$\rho^{(1)} = \frac{e^{-\frac{V^2}{2}}}{(2\pi)^{\frac{3}{2}}} \left\{ \frac{P}{2}\,(VV - I) : \nabla_x \nabla_x \phi^{(0)} + V \cdot \left[ P\, \nabla_x \left( \phi_t^{(0)} - \frac{|\nabla_x \phi^{(0)}|^2}{2} \right) - \nabla_x P \right] \right\}. \tag{69}$$

Notice that in the one-dimensional case, the term $\nabla_x \left( \phi_t^{(0)} - \frac{|\nabla_x \phi^{(0)}|^2}{2} \right)$ vanishes. We could have added a term which solves the homogeneous equation $\mathcal{L}u = 0$ in the right hand side of (69). However, as we explained before, the effect of this term is already included in $P(x,t;\epsilon)$. The solvability condition for the $\rho^{(2)}$ equation yields

$$F^{(1)} = \nabla_x \cdot \left\{ P\, \nabla_x \left( \frac{|\nabla_x \phi^{(0)}|^2}{2} - \phi_t^{(0)} \right) + \nabla_x P \right\}. \tag{70}$$

The reduced Smoluchowski equation for $P$ is therefore

$$P_t = \nabla_x \cdot \left\{ P \nabla_x \phi + \epsilon\, P\, \nabla_x \left( \frac{|\nabla_x \phi|^2}{2} - \phi_t \right) + \epsilon\, \nabla_x P \right\}, \tag{71}$$

up to terms of order $\epsilon^2$. Since $P = \nabla_x \cdot E$, $E = -\nabla_x \phi$, we may rewrite this equation as

$$E_t + E(\nabla_x \cdot E) = \epsilon\, \{ (\nabla_x \cdot E)\, [(E \cdot \nabla_x) E + E_t] + \nabla_x (\nabla_x \cdot E) \} + \mathcal{J}\,, \tag{72}$$

where $\nabla_x \cdot \mathcal{J} = 0$, so that $\mathcal{J}$ is a solenoidal field. In the one-dimensional case, $\mathcal{J} = 0$, $E_t = -E E_x + O(\epsilon)$, and (72) becomes the Burgers equation (62).

## IV. CLOSURE OF EQUATIONS FOR THE MOMENTS

Equations (61) or (62) can also be obtained by appropriate closure of the hierarchy of equations for the moments of $\rho$, as we will show now. The equations for the first two moments of $\rho$,

$$P = \int \rho\, dv, \quad J = \int v \rho\, dv,$$

can be obtained directly from the VPFP system:

$$P_t + J_x = 0, \tag{73}$$

$$-J - (\phi_x P)_x = \epsilon J_t + \epsilon\, \frac{\partial}{\partial x} \int v^2 \rho\, dv. \tag{74}$$

If we take the $x$-derivative of the second equation and eliminate $J$ by using the first one, we obtain

$$P_t - (\phi_x P)_x = -\epsilon P_{tt} + \epsilon \frac{\partial^2}{\partial x^2} \int v^2 \rho \, dv. \qquad (75)$$

Let us now close the hierarchy of equations by inserting the zeroth order ansatz (46)

$$\int v^2 \rho \, dv \sim \int v^2 \frac{e^{-\frac{V^2}{2}}}{\sqrt{2\pi}} P \, dv = (1 + \phi_x^2) \, P \qquad (76)$$

in (75). The result is the nonlinear wave equation [10]:

$$P_t - (\phi_x P)_x = \epsilon \left\{ \frac{\partial^2}{\partial x^2} \left[ (1 + \phi_x^2) \, P \right] - P_{tt} \right\}, \qquad (77)$$

to be solved together with (48) and (49). For the electric field, $E = -\phi_x$, $P = E_x$, and (77) yields

$$E_t + E \, E_x + \epsilon \, E_{tt} - \epsilon \frac{\partial}{\partial x} \left[ (1 + E^2) \, E_x \right] = 0, \qquad (78)$$

after integrating once with respect to $x$ and using the decay conditions to cancel the resulting constant. We can now show that this equation is compatible with our previously derived Burgers equation. Let us iterate (78) to get

$$\begin{aligned} E_{tt} &\sim -(E \, E_x)_t = -E_t E_x - E E_{xt} \\ &\sim E E_x^2 + (E \, E_x)_x \, E = (E^2 E_x)_x \, . \end{aligned} \qquad (79)$$

Inserting this result in (78), we recover the Burgers equation (62). In the limit as $\epsilon \to 0+$, both the Burgers equation and Equation (78) regularize shock waves and yield the same shock speed, $(E_+ + E_-)/2$, for a shock connecting the field values $E_-$ and $E_+$ [11]. However the inner structure of the shock is different in both cases and it would be interesting to study Equation (78) for its own sake.

## V. CONCLUSIONS

We have derived a reduced equation for the charge density (or the electric field) of the VPFP system in the high-field limit. The simplicity of the Fokker-Planck collision term allows us to explicitly find a zeroth order solution which is a displaced Maxwellian. This result is the basis of two classical singular perturbation procedures used to obtain the reduced equation: the Hilbert expansion and the Chapman-Enskog method. We find that the second method is ideally suited to provide a reduced equation whose terms may be of different order. In the present case, we obtain a form of the Burgers equation. On the other hand, closure of the equations for the moments of the probability density yields a nonlinear wave equation, which is compatible with the previous result, but may be of independent interest. Another interesting open problem is to use the CEM in the rigorous analysis of the high-field limit of the VPFP system. So far, rigorous analyses have used the less systematic method of closing moment equations. The trouble with using CEM seems to be that it is more "nonlinear" (albeit systematic) than the other methods.

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