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Effects of disorder on the wave front depinning transition in spatially discrete systems

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Pinning and depinning of wave fronts are ubiquitous features of spatially discrete systems describing a host of phenomena in physics, biology, etc. A large class of discrete systems is described by overdamped chains of nonlinear oscillators with nearest-neighbor coupling and subject to random external forces. The presence of weak randomness shrinks the pinning interval and it changes the critical exponent of the wave front depinning transition from 1/2 to 3/2. This effect is derived by means of a recent asymptotic theory of the depinning transition, extended to discrete drift-diffusion models of transport in semiconductor superlattices and is confirmed by numerical calculations.

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Phenomena in many different fields may be described by means of spatially discrete systems: motion of dislocations in crystals [1], atoms adsorbed on a periodic substrate [2], arrays of coupled diode resonators [3], weakly coupled semiconductor superlattices (SL) [4,5], sliding of charge density waves (CDW) [6], superconductor Josephson array junctions [7], propagation of nerve impulses along myelinated fibers [8,9], pulse propagation through cardiac cells [9], calcium release waves in living cells [10], etc. In many of these systems, the disorder due to differences in the parameters of individual elements is important because it has a strong impact on the collective behavior. A distinctive example of collective behavior in discrete systems (not shared by continuous ones) is the phenomenon of wave front pinning: for values of a control parameter in a certain interval, wave fronts joining two different constant states fail to propagate [9]. When the control parameter surpasses a threshold, the wave front depins and starts moving [8]. The existence of such thresholds is an intrinsically discrete fact, which is lost in continuum aproximations. Recently, a theory of front depinning and motion near threshold has been proposed by two of us for one-dimensional nonlinear spatially discrete reaction-diffusion systems [11]. In our theory, propagation failure and front depinning are characterized by studying the behavior of a few sites, provided the effects of spatial discretization are sufficiently strong [5,11].

In this paper, we consider the effect of weak disorder on the wave front depinning transition in spatially discrete 1D systems. Applications will include discrete RD systems subject to a random field, sliding CDW and domain motion in SL. In these examples, the main effect of disorder is to soften the transition, changing the critical exponent from 1/2 to 3/2. The latter value was obtained by Fisher in a mean-field model of sliding CDW using scaling arguments [12].

We consider chains of diffusively coupled overdamped oscillators in a potential V, subject to a random force field $F+\gamma\xi_n$

$$\frac{du_n}{dt}=u_{n+1}-2u_n+u_{n-1}+F-Ag(u_n)+\gamma\xi_n. \qquad (1)$$

Here $g(u)=V'(u)$ presents a "cubic" nonlinearity, such that $Ag(u)-F$ has three zeros, $U_1(F/A)<U_2(F/A)<U_3(F/A)$ in a certain force interval $[g'(U_i(F/A))>0$ for $i=1,3$, $g'(U_2(F/A))<0]$. The fluctuating part of the force field is $\gamma\xi_n$, where $\gamma\geqslant 0$ characterizes the disorder strength and ξ_n is a zero mean random variable taking values on an interval $(-1,1)$ with equal probability. An example of a model described by Eq. (1) is (except for the mean-field approximation, which we do not make) Fisher's modification of the Fukuyama-Lee model of sliding CDW [12]. In it, $u_n=\theta_n-\chi_n$, $g(u)=\sin u$, $\gamma\xi_n=\chi_{n+1}-2\chi_n+\chi_{n-1}$, where θ_n is the CDW phase at the site n and χ_n is a random variable taking values with equal probability on $(0,2\pi)$.

Provided $g(u)$ is odd with respect to $U_2(0)$ and $\gamma=0$, there is a symmetric interval $|F|\leqslant F_c$ where the wave fronts joining the stable zeros $U_1(F/A)$ and $U_3(F/A)$ are pinned. For $|F|>F_c$, there are *smooth traveling wave fronts*, $u_n(t)=u(n-ct)$, with $u(-\infty)=U_1$ and $u(\infty)=U_3$. The velocity $c(A,F)$ depends on A and F and it satisfies $cF<0$ and $|c|\propto(|F|-F_c)^{1/2}$ as $|F|\to F_c$ [11]. Examples are the overdamped Frenkel-Kontorova model ($g=\sin u$) [13] and the quartic double-well potential $[V=(u^2-1)^2/4]$. Less symmetric nonlinearities yield a nonsymmetric pinning interval and our analysis of the depinning transition applies to them with trivial modifications [5].

Let us recall the main features of the active point theory of the wave front depinning transition in the absence of disorder [11,5]. Except when A is too small (the continuum limit in which $F_c\to 0$), the stable wave front profile differs appreciably from either U_1 or U_3 at finitely many points, u_n, $n=-L,\dots,-1,0,1,\dots,M$, called the *active points*. At $n<-L$, $u_n\approx U_1(F/A)$ and at $n>M$, $u_n\approx U_3(F/A)$. We shall reconstruct the wave front profile $u(n-ct)$ by analyzing the behavior of the active points $u_n(t)$, $n=-L,\dots,-1,0,1,\dots,M$, as the front moves for $F>F_c>0$ (the case $F<0$ can be obtained by using symmetry considerations). The arbitrary phase of the (translation invariant) wave front will be fixed by imposing that the solution of the system of active points at time $t=0$ (and F slightly larger than F_c) be

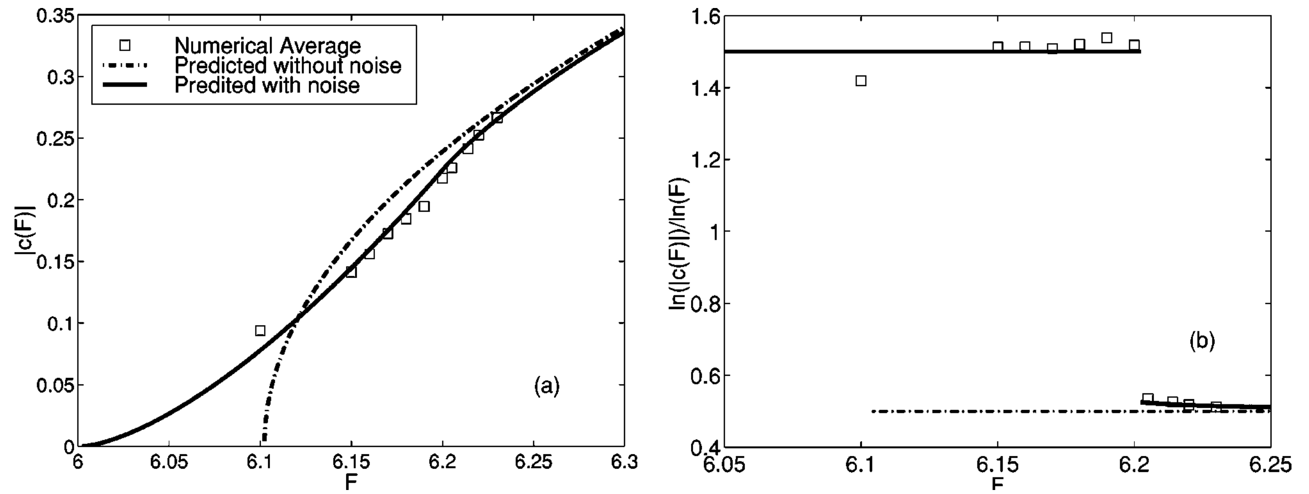

FIG. 1. (a) Average velocity $|\bar{c}|$ as a function of F for $A=10$, $F_c=6.102\,281$, and $\gamma=0.1$. (b) Graph of $\ln|\bar{c}|/\ln F$ showing the crossover to the critical exponent 3/2.

equal to its stationary solution at $F=F_c$, $u_n(A,F_c)$, $n=-L,\ldots,M$ up to terms of order $(F-F_c)$. F_c is obtained from the condition that the matrix of the coefficients in the system of active points linearized about the stationary solution has a zero eigenvalue. Provided V_n $(V_{-L}^2+\cdots+V_M^2=1)$ is the corresponding eigenvector, an outer approximation to the solution $u_n(t)$ is $u_n(t)\sim u_n(A,F_c)+\varphi(t)V_n$, where the amplitude φ obeys the equation $d\varphi/dt=\alpha(F-F_c)+\beta\varphi^2$, in which $\alpha=\Sigma_{i=-L}^{M}V_i+A^{-1}[V_{-L}/g'(U_1(F_c/A))+V_M/g'(U_3(F_c/A))]>0$, $\beta=-(A/2)\Sigma_{i=-L}^{M}g''(u_i(A,F_c))V_i^3>0$. The solution of this equation such that $\varphi(0)=0$ [equivalent to $u_n(0)=u_n(A,F_c)$ for $n=-L,\ldots,M$] is $\varphi=[\alpha(F-F_c)/\beta]^{1/2}\tan[\sqrt{\alpha\beta(F-F_c)}t]$. The amplitude φ blows up at times $t=\pm t_b$, $t_b=\pi/[2\sqrt{\alpha\beta(F-F_c)}]$. The inverse width of this time interval yields an approximation for the wave front velocity, $|c|\sim\sqrt{\alpha\beta(F-F_c)}/\pi$. At the blow-up times, the previous outer approximation to $u_n(t)$ has to be matched to an appropriate inner solution. At the later blow-up time t_b, the appropriate inner solution is the solution of the active point system at $F=F_c$ with the boundary conditions that $u_n=u_n(A,F_c)$ as $t\to-\infty$ and $u_n=u_{n+1}(A,F_c)$ as $t\to\infty$. At the earlier blow-up time $-t_b$, the inner solution obeys the same system of equations at $F=F_c$, but the boundary conditions are $u_n=u_{n-1}(A,F_c)$ as $t\to-\infty$ and $u_n=u_n(A,F_c)$ as $t\to\infty$ [14].

Effects of disorder. How does weak disorder modify this picture of the wave front depinning transition? Our main idea is to find a dominant balance of the disorder effects with nonlinearities and $(F-F_c)$ near the depinning transition. Given our active point construction sketched above, the dominant balance is struck provided $(F-F_c)=O(\gamma)$ as $\gamma\to 0$. The amplitude equation becomes

$$\frac{d\varphi}{dt}=\alpha(F-F_c)+\gamma\sum_{n=-L}^{M}V_n\xi_n+\beta\varphi^2, \tag{2}$$

and the matching condition is the same as before. The solution of Eq. (2) blows up at the end of time intervals of duration $1/|c_R|$, where

$$|c_R|=\frac{1}{\pi}\sqrt{\alpha\beta(F-F_c)+\gamma\beta\sum_{n=-L}^{M}V_{R+n}\xi_{R+n}}, \tag{3}$$

provided the argument of the square root is positive. Otherwise the motion of the wave front stops and it becomes pinned. Notice that we have chosen now $u_R(t)$ as the central active point that was distinguished with the subscript zero in our previous formulas. After the blow up, u_R jumps to $u_{R+1}(A,F_c)$, approximately, and it remains there until a time $1/|c_{R+1}|$ has elapsed. Then it jumps to $u_{R+2}(A,F_c)$ approximately, and so on.

The magnitude of interest in these systems is usually an average velocity $|c_R|$ over sufficiently many points. For example, this magnitude is proportional to the current due to a sliding CDW and it is important to know its behavior near the depinning field and the magnitude thereof. We shall argue that the average velocity is approximately given by the following equation:

$$|\bar{c}|\equiv\frac{1}{N}\sum_{R=1}^{N}|c_R|=\langle|c_R(\xi)|\rangle, \tag{4}$$

$$\langle|c_R(\xi)|\rangle\equiv\frac{1}{2\pi}\int_{-1}^{1}\{\alpha\beta(F-F_c)+\gamma\beta\sigma\xi\}_{+}^{1/2}d\xi. \tag{5}$$

Here $N\gg(L+M+1)$ is sufficiently large, $\sigma=1$, and $\{x\}_{+}^{1/2}$ is $\sqrt{x}$ if $x>0$ and zero otherwise. The idea of the proof is as follows. Let us assume that A is so large that $L=M=0$ and there is only one active point. Then the central limit theorem applied to Eq. (3) with $V_R=1$ yields Eqs. (4) and (5). Let us assume now that there are two active points. The previous argument fails because now the sum in Eq. (3) comprises two terms instead of one. Then the terms in the arithmetic mean are no longer independent: when $u_R=u_1$ for instance,

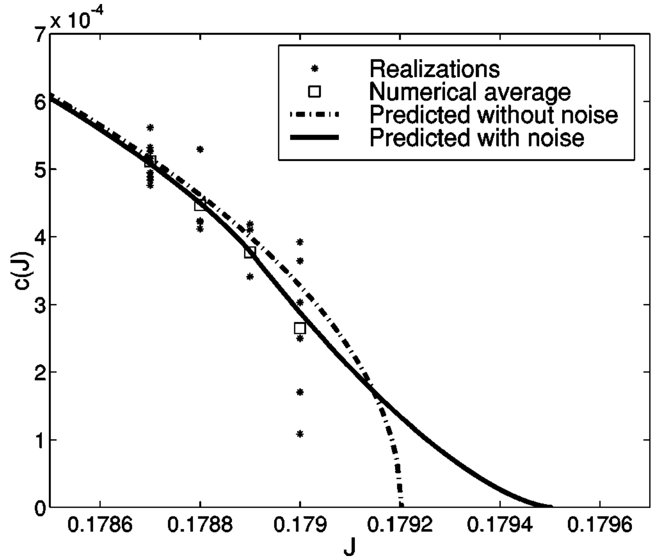

FIG. 2. Dimensionless average wave front velocity for the 9/4 SL with dimensionless parameters $\nu=3$, $J_1=0.179\,203$, $\gamma=0.01$.

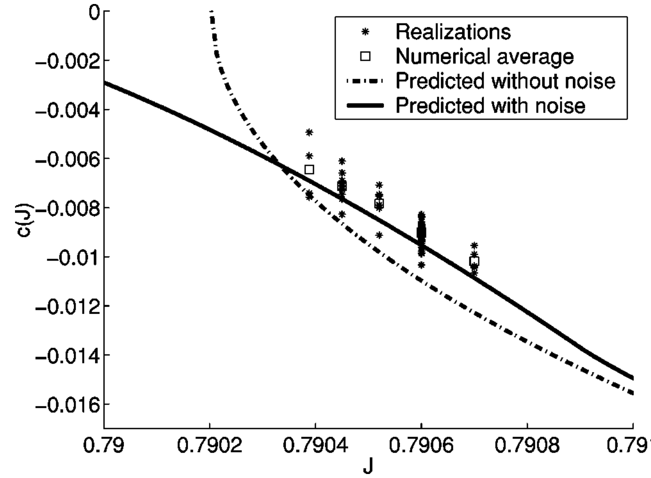

FIG. 3. Same as Fig. 2 with parameters $\nu=40$, $J_2=0.790\,203$, $\gamma=0.01$.

Eq. (3) contains ξ_1 and ξ_2. After the blow-up time, $u_R=u_2$ and Eq. (3) contains ξ_2 and ξ_3, etc. However, we can group the sums appearing in the arithmetic average of Eq. (4) in two groups containing only independent random variables: $R=2r-1$ and $R=2r$, with $r=1,2,\ldots$. The variable $V_1\xi_1+V_2\xi_2$ has zero mean and correlation $2\sigma^2/3$, where $\sigma^2=V_1^2+V_2^2=1$. This correlation is exactly the same as that of the variable ξ_1. Then the central limit theorem applied to each group (of sums of "dimer" random variables) gives one half the integral in Eq. (5), and the sum of these two halves yields Eq. (4). If we have more active points, we just have to subdivide the arithmetic mean in as many subgroups as active points and use the previous argument to prove Eq. (4).

The elementary integral in Eq. (5) yields $\overline{c}=0$ if $F<F_c-\gamma\sigma/\alpha$ (recall that $\sigma=1$),

$$|\overline{c}|=\frac{\sqrt{\beta\sigma}}{3\pi\gamma}\times\begin{cases}\left|\frac{\alpha}{\sigma}(F-F_c)+\gamma\right|^{3/2}, & \text{if } |F-F_c|\leqslant\frac{\gamma\sigma}{\alpha}\\ \left|\frac{\alpha}{\sigma}(F-F_c)+\gamma\right|^{3/2}-\left|\frac{\alpha}{\sigma}(F-F_c)-\gamma\right|^{3/2},\end{cases} \tag{6}$$

if $(F-F_c)>\gamma\sigma/\alpha$. Clearly we have a new critical field $F_c^*=F_c-\gamma\sigma/\alpha$, and a new critical exponent 3/2 (instead of 1/2 for the case without disorder), $|\overline{c}|\propto(F-F_c^*)^{3/2}$. We have compared our theory with the direct numerical solution of Eq. (1) in Fig. 1, obtaining an excellent agreement of theoretical predictions and numerical simulation.

Similarly, we can analyze the effect of weak disorder in the doping of the wells on the motion of wave fronts in dc current biased semiconductor SL. If the total current density is close to a pinning value, the displacement current is almost zero except at certain times during wave front motion at which the wave front jumps from one well to the adjacent one. The average velocity given by Eq. (4) is proportional to the arithmetic mean of time averages of the displacement current over the time interval between maxima thereof. The average velocity is basically the mean velocity at which the wave front traverses N wells. To calculate it, we take advantage of our theory of wave front motion in current biased SL [5]. The equations we use are those in Ref. [5] except that the dimensionless Poisson equation is now $E_i-E_{i-1}=\nu(n_i-1-\gamma\xi_i)$ where γ and ξ defined as in Eq. (1) represent the disorder in well doping. In the SL equations, the roles of the force F and the parameter A are taken by the total current density J and the dimensionless doping ν. If $\gamma=0$ and ν surpasses a certain minimal value, there are two critical values of the current, J_1 and J_2, such that a wave front is pinned if $J_1\leqslant J\leqslant J_2$ and it moves at a constant velocity $c(J,\nu)$ otherwise. The velocity c is positive if $J<J_1$ and negative if $J>J_2$. Furthermore, near the critical currents, $|c|\propto|J-J_c|^{1/2}$ (J_c is either J_1 or J_2) [5]. How does the disorder correct this picture? The effect of disorder is to add a term $\gamma D(E_j)\xi_{j+1}-\gamma[v(E_j)+D(E_j)]\xi_j$ to the total current density J in the dimensionless discrete Ampère equation. Then the theory in Ref. [5] yields an equation similar to Eq. (3) for the front velocity

$$|c_R|=\frac{1}{\pi}\sqrt{\alpha\beta(J-J_c)-\gamma\beta\sum_{n=-L}^{M}U_{R+n}^{\dagger}W_{R+n}}, \tag{7}$$

$$W_{R+n}=(v_{R+n}+D_{R+n})\xi_{R+n}-D_{R+n}\xi_{R+n+1}. \tag{8}$$

Here, (i) $U_{R+n}^{\dagger}$ is the left eigenvector corresponding to the zero eigenvalue of the linearized equations about the stationary solution at $J=J_c$ (chosen in such a way that $\Sigma_{n=-L}^{M}U_{R+n}^{\dagger}U_{R+n}=1$; U_{R+n} is the right eigenvector), (ii) we define $v_{R+n}=v(E_{R+n})$, etc., and (iii) α and β are given by Eqs. (20) and (21) in Ref. [5]. The values E_{R+n} are those of the stationary electric field profile at $J=J_c$. The noise term in Eq. (8) can be written as $\Sigma[U_{R+n}^{\dagger}(v_{R+n}+D_{R+n})-U_{R+n-1}^{\dagger}D_{R+n-1}]\xi_{R+n}$, provided we take $U_{R+n}^{\dagger}=0$ for $n<-L$ and $n>M+1$. The noise term has zero average and a correlation $2\sigma^2/3$, with $\sigma^2=\Sigma[U_{R+n}^{\dagger}(v_{R+n}+D_{R+n})-U_{R+n-1}^{\dagger}D_{R+n-1}]^2$. Using the previously mentioned argument of splitting the arithmetic mean in groups of independent random variables, we can show that Eqs. (4) and (5) hold provided σ in Eq. (5) is given by the previous formula and $(F-F_c)$ is replaced by $(J-J_c)$. Comparisons of the resulting average front velocity $\overline{c}$ with the results of numerically solving the SL model are shown in Figs. 2 and 3 for currents close to J_1 and J_2, respectively.

In conclusion, we have shown that weak disorder changes qualitatively the wave front depinning transition in overdamped one-dimensional discrete models. Disorder shrinks the pinning interval and it changes the critical exponent for the velocity from 1/2 to 3/2. Whether these features are robust and hold for strong disorder remains to be seen. An interesting indication is that the critical exponent 3/2 is obtained independently of the noise strength in mean-field models of sliding CDW [12].

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