A Thin Cantilever Beam in a Flow

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Abstract. We study the small vibrations of a thin flexible beam immersed in a laminar flow, in which we assume the dominant restoring force in most of the domain is tension due to the shear stress, while bending elasticity plays a small but non-negligible role. A linearized description is considered, which is reduced to an eigenvalue problem. The resulting singularly-perturbed problem is solved asymptotically up to the first modification of the eigenvalue.

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INTRODUCTION

Model linear transverse vibrations of a thin rod in a laminar flow are modeled as being governed by the tension due to the shear stress and by the bending rigidity of the rod. The interaction between the fluid and the beam is modeled by assuming that the tension in the rod increases linearly from the free tip of the rod, at $y = 0$, to where it is clamped, at $y = L$. Specifically, the tension of the rod is assumed to be given by $T(y) = 2(a+b)\mu \frac{dU}{dn} y$, where a and b are the width and thickness of the beam, μ is the dynamic viscosity of the liquid and $\frac{dU}{dn}$ is the shear velocity. The governing equation for the transverse deflections of the rod is therefore

$$\rho w_{tt} - 2(a+b)\mu \frac{dU}{dn}(y w_y)_y + \frac{Eab^3}{12} w_{yyyy} = 0, \tag{1}$$

where E is Young's modulus and the term $\frac{ab^3}{12}$ is the cross-sectional moment of rectangular beam. Equation (1) is studied in the asymptotic limit where the bending forces are small compared to tension, using singular perturbations.

Equation (1) is supplemented with two boundary conditions at each end of the domain. These are the standard clamped/free boundary conditions:

$$w_{yy}(0,t) = 0, \quad w_{yyy}(0,t) = 0, \quad w(L,t) = 0, \quad w_y(L,t) = 0. \tag{2}$$

We scale y by the length L and t using the tension-wave speed: $\sqrt{\rho L / 2(a+b)\mu \frac{dU}{dn}}$. This results in

$$w_{tt} - (y w_y)_y + \varepsilon w_{yyyy} = 0, \qquad \text{where } \varepsilon = \frac{Eab^3}{24\mu(a+b)L^3}. \tag{3}$$

The dimensionless constant ε compares the importance of bending elasticity ($\mathscr{O}(Eab^3/12L^2)$) with the maximum tension ($\mathscr{O}(2(a+b)\mu \frac{dU}{dn} L)$). Notice that ε depends on the geometry, but not the scaling of the beam, it remains unchanged if L, a, and b are scaled together.

A solution of (3) in separated form $w(y,t) = u(y) \times \Omega(t)$, implies that u must satisfy

$$\varepsilon u'''' - (y u')' = \lambda u, \qquad 0 < y < 1, \tag{4}$$

where λ is the eigenvalue parameter, and

$$\Omega(t) = \Omega_0 e^{\pm i \sqrt{\lambda} t}. \tag{5}$$

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Equation (4) is singular for two reasons: (i) ε multiplies the highest-order derivative in the equation and (ii) the coefficient of leading-order derivative in the reduced equation (after setting $\varepsilon = 0$), vanishes at one end of the interval. in the following this *singularly perturbed* eigenvalue problem is solved, subject to the boundary conditions

$$u''(0) = 0, \quad u'''(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \quad u(0) = 1, \tag{6}$$

last condition having been added for normalization purposes.

A comparison of the first eigenfunction to the numerical and leading-order approximation is shown in Fig. 1(a).

PREPARATION FOR THE ASYMPTOTIC SOLUTIONS

We expect there to be two boundary layers connecting the solution in the bulk to the boundaries Near the boundaries, we expect that the fourth derivative becomes large enough that, even when multiplied by ε, it cannot be ignored.

As a first step we find the appropriate boundary-layer scalings.

The boundary at $y = 0$ is governed by a balance between the two terms on the LHS of (4), resulting in the scaling of y as $y = X\varepsilon^{-1/3}$ and $u(y;\varepsilon) \approx U(X;\varepsilon)$. The boundary layer equation is

$$U'''' - (XU')' = \varepsilon^{1/3}\lambda U. \tag{7}$$

The boundary at $y = 1$ is governed by the same two terms, however the resulting balance gives $y = 1 - \varepsilon^{-1/2}$, $u(y;\varepsilon) \approx V(Z;\varepsilon)$, and

$$V'''' - \left(\left(1 - \varepsilon^{1/2}Z\right)V'\right)' = \varepsilon\lambda V. \tag{8}$$

In the asymptotic series for the solution, terms of order 1/3 and 1/2 are forced by the boundary layers, and subsequent terms include all sums of integer multiples of 1/3 and 1/2* Thus, we look for a solution that has the following form:

- In the bulk we expand the solution $u(y;\varepsilon)$ as

$$u(y;\varepsilon) = u_0(y) + \varepsilon^{1/3}u_{1/3}(y) + \varepsilon^{1/2}u_{1/2}(y) + \varepsilon^{2/3}u_{2/3}(y) + \varepsilon^{5/6}u_{5/6}(y) + \varepsilon u_1(y) + \dots \tag{9}$$

- Near the boundary, the solution is approximated by a similar asymptotic series for $U(X;\varepsilon)$, near $y = 0$ and for $V(Z;\varepsilon)$, near $y = 1$.
- The eigenvalue† λ is itself expanded as an asymptotic series:

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon^{1/3}\lambda_{1/3} + \varepsilon^{1/2}\lambda_{1/2} + \varepsilon^{2/3}\lambda_{2/3} + \varepsilon^{5/6}\lambda_{5/6} + \varepsilon\lambda_1 + \dots \tag{10}$$

For the three domains, we substitute the asymptotic expansion of u, U, or V, and of λ into the appropriate differential equation and collect powers of ε. Doing this for an expansion up to ε^1 results in

$$(yu_i')' + \lambda_0 u_i = -\sum_{j>0} \lambda_j u_{i-j}, \qquad \text{for } u_i, \tag{11}$$

$$U_i'''' - (XU_i')' = \sum_{j>0} \lambda_j U_{i-j-1/3}, \qquad \text{for } U_i, \text{ and} \tag{12}$$

$$V_i'''' - V_i'' = \lambda_{i-1}V_{i-1} - (Z(V_{i-1/2})')', \qquad \text{for } V_i. \tag{13}$$

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* At orders equal to or greater than 1, logarithmic terms enter as well, but we will not get to that here.

† Of course, there is a sequence of eigenvalues with corresponding eigenfunctions. We drop the superscript in the $\lambda^{(n)}$ notation for sake of a cleaner presentation.

In shorthand above, a term containing something that does not exist (like $\lambda_{-1/2}$, or $\lambda_{1/6}$) should be ignored, so the RHS for $i = 1/3$ of the equations is $-\lambda_{1/3}u_0$, $\lambda_0 U_0$, and 0.

The general form of solution of the associated homogeneous equation of the reduced equations in the bulk (11) is given by a linear combination of

$$\tilde{J}(y) = J_0(2\sqrt{\lambda_0 y}) \qquad \text{and} \qquad \tilde{Y}(y) = \pi Y_0(2\sqrt{\lambda_0 y}) - (\log\lambda_0 + 2\gamma)\tilde{J}(y) \tag{14}$$

where J_0 and Y_0 are the zeroth Bessel functions of the first and second kinds, and γ is Euler's constant. This non-standard basis of the Bessel equation insures that the asymptotic behavior of $\tilde{Y}(y)$ as $y \to 0$ is simply

$$\tilde{Y}(y) = \log(y) + \mathscr{O}(y\log y). \tag{15}$$

In the inhomogeneous case we have

Lemma 1 *Given a continuous function $g(y)$, there exists exactly one solution of $(yw')' + \lambda_0 w = g(y)$, that is continuous on the closed interval $[0,1]$ and satisfies $w(0) = 0$.*

Corollary 2 *If λ_0 satisfies $J_0(2\sqrt{\lambda_0}) = 0$ and $g(y) = \tilde{J}(y)$, then the solution to Lemma 1, $w(y)$, does not vanish at $y = 1$.*

From this point we let $w(y)$ be defined by the unique function that satisfies Corollary 2.

The solution to the associated homogeneous equation of (12) is spanned by

$$U^{(1)} = 1, \qquad U^{(2)}(X) = \int_0^X \mathrm{Ai}(x)dx, \qquad U^{(3)}(X) = \int_0^X \mathrm{Bi}(x)dx, \qquad U^{(4)} = \Psi(X) \tag{16}$$

where $\Psi(x)$ satisfies

$$\Psi''' - X\Psi' = -1. \tag{17}$$

The existence of such a function is given by

Lemma 3 *There is a unique solution $\Psi(X)$ of (17) such that*

$$\textit{(a)}\ \Psi(0) = 0, \quad \textit{(b)}\ \Psi''(0) = 0 \quad \textit{and} \qquad \textit{(c)}\ \Psi(X) = \log(X) + \mathscr{O}(1)\ \textit{as}\ X \to \infty.$$

ASYMPTOTIC SOLUTION

We calculate terms in the asymptotic seriesup to the first non-trivial correction to the eigenvalue: i.e., order $1/2$. The results are summarized in Table 1.

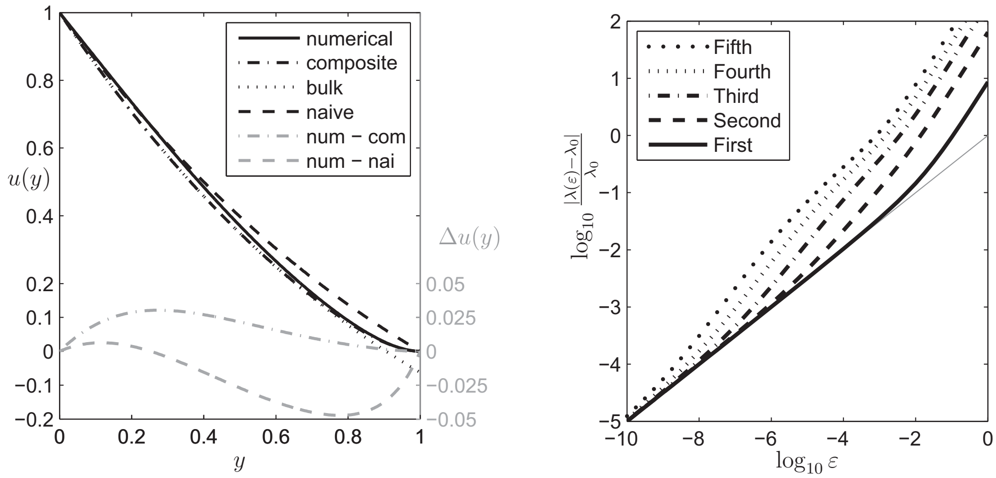
TABLE 1. Coefficients of the asymptotic expansion of the solution.

α	λ_α	u_α	U_α	V_α
0	λ_0	$\tilde{J}(y)$	1	0
$\frac{1}{3}$	0	0	$-\lambda_0 X$	0
$\frac{1}{2}$	λ_0	$-\lambda_0 w(y)$	0	$\tilde{J}'(1)\left[1 - Z - e^{-Z}\right]$

At the order ε^0 the boundary conditions narrow down the behavior at the boundary layers to be

$$U_0(X) \equiv 1, \qquad V_0(Z) = A(1 - Z - e^{-Z}). \tag{18}$$

Where $A = 1$ is found by matching with the bulk solution, which also narrows down the bulk solution to $\tilde{J}(x)$, and λ_0 must be a root of $J_0(2\sqrt{\lambda_0})$.



(a) Eigenfunction.

(b) Eigenvalue.

FIGURE 1. (a) Comparison of numerical, naive, bulk, and composite approximations of order 1/2 to the first eigenfunction, for $\varepsilon = 10^{-2}$. Differences are overlaid for clarification. The label "naive" refers to the leading order approximation. (b) A log-log plot of the relative error $e(\varepsilon) = \left|\frac{\lambda^{(n)}(\varepsilon)-\lambda^{(n)}(0)}{\lambda^{(n)}(0)}\right|$ for the first five eigenvalues and a reference line of slope 1.

The next order, $\varepsilon^{1/3}$, is similarly straightforward: The boundary conditions imply that

$$U_{1/3}(X) = -\lambda_0 X, \qquad V_{1/3}(Z) = A(1 - Z - e^{-Z}). \tag{19}$$

However, here matching (with use of Corrolary 2) implies that $A = 0$ and $u_{1/3} = 0$, and that $\lambda_{1/3} = 0$.

At the $\varepsilon^{1/2}$ order a non zero correction to the eigenvalue appears. Here, the boundary conditions imply that

$$U_{1/2} \equiv 0 \quad \text{and} \quad V_{1/2}(Z) = A(1 - Z - e^{-Z}). \tag{20}$$

While the solution in the bulk is of the form $u_{1/2}(y) = -\lambda_{1/2} w(y) + a\tilde{J}(y) + b\tilde{Y}(y)$ Matching at $y = 0$ implies that $a = b = 0$ and thus $u_{1/2}(y) = -\lambda_{1/2} w(y)$. At $y = 1$ the matching implies that $\lambda_{1/2} = -\tilde{J}'(1)/w(1)$. Incidentally, we find that $\lambda_{1/2} = \lambda_0$, due to a property of the function $w(y)$.

In Figure 1(b) a log-log plot of the error in the two-term approximation $\lambda_0 + \lambda_{1/2}\varepsilon^{1/2}$ for the eigenvalues is shown. The resulting lines of slopes near 1 suggest that the next non-vanishing correction to the eigenvalue will happen at the $\mathcal{O}(\varepsilon)$ level.

CONCLUSIONS

The purely numerical solution to this singular problem is both uninspiring and difficult to attain. The asymptotic expansion shows that the first correction to the solution scales like $\sqrt{\varepsilon}$, and that the correction to the derivative is of order 1. For higher derivatives, more terms in the asymptotic series will be required in order to extract the correct leading order behavior. For applications where the derivatives are important, this study provides a glimpse at the solution and its dependence on ε.