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HIGHER ORDER AVERAGING OF LINEAR FOKKER–PLANCK EQUATIONS WITH PERIODIC FORCING*

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Abstract. We investigate linear Fokker–Planck equations or stochastic Hamiltonian systems with periodic forcing where the impact of deterministic forcing is not captured by classical stochastic averaging. To overcome this problem, a formal energy projection method is introduced, which splits the corresponding Fokker–Planck equation and allows the computation of higher order stochastic averages. Generally, the resulting averages have to be computed with a numerical scheme. We illustrate the results of our method with two examples: the linear oscillator with periodic forcing and a nonlinear oscillator.

Key words. Fokker–Planck equations, stochastic Hamiltonian system, periodic driving forces, higher order averaging

AMS subject classifications. 37H10, 60H30, 70H05

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1. Introduction.

In the present paper a kinetic Fokker–Planck equation with periodic forcing or, equivalently, a periodically driven stochastic Hamiltonian system is investigated for small values of forcing and noise. These systems have received a lot of attention in different fields of applications; see, e.g., [7]. By classical stochastic averaging (see [8]), a limit equation for the energy dynamics can be derived for the case of small noise. Corresponding higher order averaging procedures have been presented and applied to simple explicitly computable cases in [9, 10]. The goals of the present paper are to apply a high order averaging methodology to more complicated cases, including general stochastic Hamiltonian systems with periodic forcing, and to obtain explicit or numerically computed coefficients for the limit equations. In particular, the influence of the forcing on the limiting energy equation is investigated. To this end a formal projection procedure is developed. First we apply standard stochastic averaging to the system, and then we present and analyze our formal ansatz for higher order averaging based on the Fokker–Planck equation. In general, for example for nonlinear problems, the higher order coefficients cannot be computed explicitly but have to be determined numerically. It will turn out that, in many cases, the zeroth- and first order ansatz yields the same equation for the energy as the nonforced case, since the influence of periodic forcing on the system is not visible in the limit equations. However, the influence of the forcing is captured to second order. Determining the coefficients requires the development of a numerical procedure to evaluate averages over orbits of the Hamiltonian system of first- and second order derivatives of numerically determined functions, which has to be done in a sufficiently accurate way.

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The obtained coefficients are then finally used to compare the solutions of the reduced energy equations numerically with the solutions of the full system.

We mention that a similar problem is appearing for fiber lay down in a textile production process. In [6] a stochastic process describing fiber lay down has been investigated for the small noise (small turbulence) case applying stochastic averaging. If the motion of the conveyor belt is included in this model, it turns out that the influence of the belt motion can be rewritten as a periodic forcing term acting on a suitable Hamiltonian stochastic system; see [4]. The classical stochastic averaging procedure will not yield any influence of the belt motion on the limit equation. Such an influence will be captured only by a procedure similar to that presented here for the classical case of periodically driven stochastic systems.

The paper is organized as follows. In section 2, we present the kinetic Fokker–Planck model. The classical method of stochastic averaging to zeroth order for small noise and small forcing is introduced in section 3. In section 4 a formal projection procedure for higher order averaging based on the Fokker–Planck equation is presented with computations for the first- and second order cases. Finally, the numerical methods are described and numerical results are presented in section 5 for a linear and a nonlinear example.

2. The model. We study a linear kinetic Fokker–Planck equation with periodic forcing,

$$\partial_t f + \partial_v H \partial_x f - \partial_x H \partial_v f - \kappa \partial_v (g \cos\phi f) + \partial_\phi (cf) = \frac{A^2}{2} \partial_v (k^2 (\partial_v f + f \partial_v H)), \tag{2.1}$$

with Hamiltonian function $H(x,v) = V(x) + W(v)$, $g = g(x,v)$, $c = c(x,v)$, $k = k(x,v)$, $A \in \mathbb{R}$, $\kappa \in \mathbb{R}$, or, equivalently (see [13]), the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}x &= \partial_v H \, \mathrm{d}t, \\ \mathrm{d}v &= -\partial_x H \, \mathrm{d}t - \kappa g \cos\phi \, \mathrm{d}t - \frac{A^2}{2} (k^2 \partial_v H - \partial_v k^2) \, \mathrm{d}t + Ak \, \mathrm{d}W_t, \\ \mathrm{d}\phi &= c \, \mathrm{d}t. \end{aligned} \tag{2.2}$$

We focus on the regime of *small noise* A and *small forcing* κ, which means that the Hamiltonian motion is dominant and will be scaled by a factor of $1/\epsilon$. We are interested in the energy distribution of (2.1) and its deviation from the distribution of the unforced system ($\kappa = 0$). Recall that the stationary distribution for $\kappa = 0$ reads

$$p_S(x,v) = C \, e^{-H(x,v)}, \tag{2.3}$$

which is independent of A. We assume a periodic motion in phase space for the deterministic system with $\kappa, A = 0$. The general results will be illustrated with two fundamental examples.

Example 1 (the linear oscillator). Set $H = \frac{v^2}{2} + \frac{x^2}{2}$, $g = 1$, $c = \omega$, and $k = 1$ and obtain

$$\partial_t f + v \partial_x f - x \partial_v f - \kappa \partial_v (\cos\phi f) + \partial_\phi (\omega f) = \frac{A^2}{2} \partial_v (\partial_v f + v f) \tag{2.4}$$

or

$$\begin{aligned} \mathrm{d}x &= v \, \mathrm{d}t, \\ \mathrm{d}v &= -x \, \mathrm{d}t - \kappa \cos\phi \, \mathrm{d}t - \frac{A^2}{2} v \, dt + A \, \mathrm{d}W_t, \\ \mathrm{d}\phi &= \omega \, \mathrm{d}t. \end{aligned} \tag{2.5}$$

The deterministic period of motion is $T_h = 2\pi$.

Example 2 (the nonlinear oscillator). Set $H = \frac{v^2}{2} + V(x)$, where $V(x) = \frac{x^{2n}}{2n}$, $n \in \mathbb{N}$, and $c := c(H(x,v))$, and let g, k be as in Example 1. The deterministic period of motion for an energy $H(x,v) = h$ is $T_h = \frac{T_1}{h^{\frac{1}{2}-\frac{1}{2n}}}$, where T_1 is the period of motion with energy equal to 1.

The linear oscillator will be our reference example, since explicit computations are possible. The nonlinear oscillator serves as an example requiring numerical computations.

REMARK 1. *We can rewrite* (2.3) *in terms of the energy variable* H *as* $p_S(h) = \frac{\mathrm{d}}{\mathrm{d}h}\int_{H(x,v)<h} p_S(x,v)\,\mathrm{d}x\,\mathrm{d}v = C\,e^{-h}\frac{\mathrm{d}}{\mathrm{d}h}\int_{H(x,v)<h}\mathrm{d}x\,\mathrm{d}v = C\,e^{-h}T_h$ *because the period of motion of the deterministic unforced system is* $T_h = \frac{\mathrm{d}}{\mathrm{d}h}\int_{H(x,v)<h}\mathrm{d}x\,\mathrm{d}v$ *(see* [3]*).*

3. Classic stochastic averaging. In this section, we revisit results from the stochastic averaging theory presented in [8] or [12]. Consider process (2.2) without forcing ($\kappa = 0$) with small noise $A = \sqrt{\epsilon}\tilde{A}$ on associated long "time" scales $t = \tilde{t}/\epsilon$ with $0 < \epsilon \ll 1$. Dropping the tildes, the rescaled system reads

$$\begin{aligned} \mathrm{d}x &= \frac{1}{\epsilon}\partial_v H\,\mathrm{d}t, \\ \mathrm{d}v &= -\frac{1}{\epsilon}\partial_x H\,\mathrm{d}t - \frac{A^2}{2}(k^2\partial_v H - \partial_v k^2)\,\mathrm{d}t + Ak\,\mathrm{d}W_t. \end{aligned} \tag{3.1}$$

Applying Ito's formula for the energy process $H_t^\epsilon = H(x_t^\epsilon, v_t^\epsilon)$, we obtain the equation

$$\mathrm{d}H_t^\epsilon = \frac{A^2}{2}k\left(k\partial_{vv}H - k(\partial_v H)^2 + 2\partial_v k\partial_v H\right)\mathrm{d}t + Ak\partial_v H\mathrm{d}W_t. \tag{3.2}$$

The stochastic averaging theorem (see [8, 12] or [1, 2] for an application to stochastic Hamiltonian systems) yields a simplified approximation of the dynamics as $\epsilon \to 0$: The limit process for H_t^ϵ is termed H_t^0 and given by $\mathrm{d}H_t^0 = a(H_t^0)\mathrm{d}t + \sigma(H_t^0)\,\mathrm{d}W_t$ with drift and variance

$$a(h) = \frac{A^2}{2}\overline{k\left(k\partial_{vv}H - k(\partial_v H)^2 + 2\partial_v k\partial_v H\right)}, \tag{3.3a}$$

$$\sigma^2(h) = A^2\overline{k^2(\partial_v H)^2}. \tag{3.3b}$$

Here, the *time average* of a function f is defined as

$$\overline{f} = \overline{f}(h) = \frac{1}{T_h}\int_0^{T_h} f(x_t, v_t)\mathrm{d}t, \tag{3.4}$$

where x_t, v_t are *unforced* ($\kappa = 0$) and *deterministic* ($A = 0$) solutions of (3.1) at constant energy h and period length T_h.

REMARK 2. *Considering system* (2.2) *with periodic forcing and rescaling as above,* $A = \sqrt{\epsilon}\tilde{A}$, $t = \tilde{t}/\epsilon$, *and additionally* $\kappa = \epsilon\tilde{\kappa}$, *we obtain*

$$\begin{aligned} \mathrm{d}x &= \frac{1}{\epsilon}\partial_v H\,\mathrm{d}t, \\ \mathrm{d}v &= -\frac{1}{\epsilon}\partial_x H\,\mathrm{d}t - \kappa g\cos\phi\,\mathrm{d}t - \frac{A^2}{2}(k^2\partial_v H - \partial_v k^2)\,\mathrm{d}t + Ak\,\mathrm{d}W_t, \\ \mathrm{d}\phi &= \frac{1}{\epsilon}c\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

We note that the period of the force $\kappa g \cos\phi$ differs from that of the unforced Hamiltonian motion T_h in nonresonant situations. In this case the above averaging procedure can be performed using time averages of the form

$$\bar{f}(h,t) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x_t, v_t, \phi_t)\mathrm{d}t. \tag{3.6}$$

However, the resulting limit equations do not change compared to the unforced case, since the periodic forcing terms average to zero due to the Riemann–Lebesgue lemma. There is no influence of the forcing on the limit problem. Our aim in the following sections is to construct a higher order averaging procedure capturing the influence of the periodic forcing.

REMARK 3. *The above argument is true only as long as the situation is not resonant. Resonance appears, for example, in the case of the linear example, if $\omega = 1$; i.e., the period of the forcing is the same as the period of the deterministic motion.*

4. Higher order averaging of Fokker–Planck equations. In this section, we consider the Fokker–Planck equation (2.1) in order to develop higher order averaging procedures instead of the system of stochastic differential equations (2.2); see also [9, 10]. We describe an approach based on an energy projection method. This will yield in special cases explicit formulas for the averaged problems; however, in general the asymptotic theory has to be combined with a numerical approach to compute the limit coefficients. This section is divided as follows: We introduce the projection operators and split the Fokker–Planck equation into an equivalent system of two equations in section 4.1. A formal second order asymptotic expansion is presented in 4.2. The resulting equations are further approximated by choosing a simplifying assumption in section 4.3. The coefficients of the approximated system are then determined explicitly or numerically to various orders, and the corresponding higher-order coefficients are presented.

4.1. Resulting set of exact equations. Rescaling the Fokker–Planck equation (2.1) using $A = \sqrt{\epsilon}\tilde{A}$, $t = \tilde{t}/\epsilon$, and $\kappa = \epsilon\tilde{\kappa}$ gives

$$\partial_t f + \frac{1}{\epsilon}\partial_v H\partial_x f - \frac{1}{\epsilon}\partial_x H\partial_v f - \kappa\partial_v(g\cos\phi f) + \frac{1}{\epsilon}\partial_\phi(cf) = \frac{A^2}{2}\partial_v(k^2(\partial_v f + f\partial_v H)). \tag{4.1}$$

Following [10] and defining

$$\begin{aligned} \mathcal{L}_2 f &= -\partial_v H\partial_x f + \partial_x H\partial_v f - \partial_\phi(cf), \\ \mathcal{L}_1 f &= \kappa\partial_v(g\cos\phi f) + \frac{A^2}{2}\partial_v(k^2(\partial_v f + f\partial_v H)), \end{aligned} \tag{4.2}$$

(4.1) can be rewritten as $\partial_t f = \frac{1}{\epsilon}\mathcal{L}_2 f + \mathcal{L}_1 f$. To formulate the higher order averaging ansatz, let us start with some definitions. We define the *energy average* of functions in phase space (see [15]),

$$p(h,t) := \langle f\rangle := \partial_h \int_{H(x,v)\le h} f(x,v,\phi,t)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi, \tag{4.3}$$

which we formally write as $p(h,t) = \int \delta(H(x,v)-h)f(x,v,\phi,t)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi$. The corresponding *energy projection* from functions in phase space onto functions of energy is

$$(Pf)(x,v,t) := \frac{p(H(x,v),t)}{2\pi\Omega(H(x,v))}, \quad \text{where } \Omega(h) := \frac{\langle 1\rangle}{2\pi} = \int \delta(H(x,v)-h)\mathrm{d}x\mathrm{d}v. \tag{4.4}$$

Using the equality of energy average and time average, which is true since the ergodic theorem holds trivially for the periodic deterministic unforced Hamiltonian system of (2.2) on a sphere of constant energy, we have

$$Pf = \overline{f} = \frac{1}{T_h}\int_0^{T_h} f(x_s, v_s, \phi_s)\mathrm{d}s, \quad \overline{f} = \frac{\langle f\rangle}{2\pi\Omega}. \tag{4.5}$$

The distribution function f is then decomposed into

$$f(x,v,\phi,t) = Pf(x,v,t) + \epsilon Qf(x,v,\phi,t) \quad \text{with } \overline{Qf} = 0, \tag{4.6}$$

where the scaling factor ϵ indicates that the Hamiltonian motion dominates small forcing and small noise terms.

REMARK 4. *We note that $T_h = \Omega(h)$ (compare Remark 1), and therefore the stationary solution of the unforced case $\kappa = 0$, $p_S(h) = Ce^{-h}T_h$, fulfills $-\frac{p_s}{\Omega} = \partial_h(\frac{p_s}{\Omega})$ and $-Pf = \partial_h Pf$. Moreover, we have $\int p(h,t)\mathrm{d}h = 1\,\forall t$.*

To obtain a differential equation in terms of the energy H, we formally apply the energy average (4.3) on the Fokker–Planck equation (4.1) and insert the splitting (4.6). The result is subtracted from (4.1) to obtain a differential equation for Qf. Since $\langle \mathcal{L}_2 f\rangle = 0$, we get

$$\partial_t p - \langle \mathcal{L}_1 Pf\rangle = \epsilon \langle \mathcal{L}_1 Qf\rangle, \tag{4.7a}$$

$$\epsilon\partial_t Qf = \partial_t f - \partial_t \frac{p}{2\pi\Omega} = \left(\frac{1}{\epsilon}\mathcal{L}_2 + \mathcal{L}_1\right)(Pf + \epsilon Qf) - \frac{1}{2\pi\Omega}\langle \mathcal{L}_1(Pf+\epsilon Qf)\rangle. \tag{4.7b}$$

We compute the occurring averages in the following result.

LEMMA 4.1. *Define $S(x,v) := k\left(k\partial_{vv}H - k(\partial_v H)^2 + 2\partial_v k\partial_v H\right)$. Then*

$$\langle \mathcal{L}_1 Pf\rangle = -\frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) + \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2k^2}p\right), \tag{4.8}$$

$$\langle \mathcal{L}_1 Qf\rangle = \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2k^2Qf\right\rangle - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)Qf\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi Qf\right\rangle. \tag{4.9}$$

Proof.

$$\langle \mathcal{L}_1 f\rangle = \int \delta(H(x,v)-h)\mathcal{L}_1 f\mathrm{d}x\mathrm{d}v\mathrm{d}\phi = \int \delta(H(x,v)-h)\kappa\partial_v(g\cos\phi f)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi$$
$$+\int \delta(H(x,v)-h)\frac{A^2}{2}\partial_v(k^2(\partial_v f + f\partial_v H))\mathrm{d}x\mathrm{d}v\mathrm{d}\phi. \tag{4.10}$$

Performing integration by parts and using the chain rule gives

$$\begin{aligned}\langle \mathcal{L}_1 f\rangle = &-\int \partial_H\delta(H(x,v)-h)\cdot H_v\kappa g\cos\phi f\mathrm{d}x\mathrm{d}v\mathrm{d}\phi\\ &-\frac{A^2}{2}\int \partial_H\delta(H(x,v)-h)\cdot H_v k^2\left(\partial_v f + f\partial_v H\right)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi.\end{aligned} \tag{4.11}$$

Integrating by parts and using the chain rule again yields

$$\begin{aligned}\langle \mathcal{L}_1 f\rangle = &-\int \partial_H\delta(H(x,v)-h)\cdot H_v\kappa g\cos\phi f\mathrm{d}x\mathrm{d}v\mathrm{d}\phi\\ &+\frac{A^2}{2}\int \partial_H\delta(H(x,v)-h)\cdot\left(H_v\partial_v k^2 + H_{vv}k^2 - (H_v)^2k^2\right)f\mathrm{d}x\mathrm{d}v\mathrm{d}\phi\\ &+\frac{A^2}{2}\int \partial_H^2\delta(H(x,v)-h)\cdot H_v^2k^2 f\mathrm{d}x\mathrm{d}v\mathrm{d}\phi.\end{aligned} \tag{4.12}$$

The same arguments yield in a straightforward way $\int \delta(H(x,v)-h)\mathcal{L}_2 f \mathrm{d}x\mathrm{d}v\mathrm{d}\phi = 0$. We replace $\partial_H \delta(H(x,v)-h) = -\partial_h \delta(H(x,v)-h)$. Inserting now $f = Pf + \epsilon Qf$, we obtain, using the above definition of $S = S(x,v)$,

$$\begin{aligned}\langle \mathcal{L}_1 f\rangle &= \partial_h^2 \int \delta(H(x,v)-h)\cdot \frac{A^2}{2} H_v^2 k^2 \cdot (Pf+\epsilon Qf)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi \\ &+ \partial_h \int \delta(H(x,v)-h)\left[H_v \kappa g\cos\phi - \frac{A^2}{2}S(x,v)\right](Pf+\epsilon Qf)\mathrm{d}x\mathrm{d}v\mathrm{d}\phi.\end{aligned} \tag{4.13}$$

For the integrals over Pf, we write Pf outside the integral, since it depends on x, v only through $H(x,v)=h$. The resulting expression is

$$\begin{aligned}\langle \mathcal{L}_1 f\rangle = &-\frac{A^2}{2}\partial_h(\overline{S(x,v)}p) + \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) + \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 Qf\right\rangle \\ &- \epsilon\frac{A^2}{2}\partial_h\left\langle S(x,v)Qf\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi Qf\right\rangle.\end{aligned} \tag{4.14}$$

Here we have used $\overline{e^{i\phi} f(x,v)} = 0$. □

Inserting the results from Lemma 4.1 into (4.7), we obtain a system of exact equations, whose solution is equivalent to a solution of the Fokker–Planck equation (2.1):

$$\begin{aligned}\partial_t p + \frac{A^2}{2}\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) &= +\epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 Qf\right\rangle \\ &- \epsilon\frac{A^2}{2}\partial_h\left\langle S(x,v)Qf\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v\cos\phi Qf\right\rangle,\end{aligned} \tag{4.15a}$$

and taking $\mathcal{L}_2 Pf = 0$ into account gives

$$\begin{aligned}&\epsilon\partial_t Qf + H_v\partial_x Qf - H_x\partial_v Qf + \partial_\phi(cQf) \\ &- \epsilon\kappa\cos\phi\partial_v(gQf) - \epsilon\frac{A^2}{2}\partial_v(k^2(\partial_v Qf + Qf\partial_v H)) \\ &+ \epsilon\frac{A^2}{4\pi\Omega}\partial_h^2\left\langle H_v^2 k^2\, Qf\right\rangle + \epsilon\frac{\kappa}{2\pi\Omega}\partial_h\left\langle\left(H_v g\cos\phi - \frac{A^2}{2}S(x,v)\right)Qf\right\rangle \\ &= \frac{\kappa}{2\pi}\left(H_v g\cos\phi\partial_h\frac{p}{\Omega} + g_v\cos\phi\frac{p}{\Omega}\right) \\ &+ \frac{A^2}{4\pi}\left((k_v^2 H_v + k^2 H_{vv})\frac{p}{\Omega} + (k_v^2 H_v + k^2 H_v^2 + k^2 H_{vv})\partial_h\frac{p}{\Omega} + k^2 H_v^2\partial_h^2\frac{p}{\Omega}\right) \\ &+ \frac{A^2}{4\pi}\left(\frac{1}{\Omega}\partial_h(\overline{S}p) - \frac{1}{\Omega}\partial_h^2(\overline{H_v^2 k^2}p)\right).\end{aligned} \tag{4.15b}$$

The advantage of rearranging (2.1) into (4.15) is explained in the next section, where we derive a higher order approximation of the energy process.

Example 1 (continued). For the linear oscillator, we obtain the system

$$\begin{aligned}\partial_t p + \frac{A^2}{2}\partial_h((1-h)p) - \frac{A^2}{2}\partial_h^2(hp) &= \epsilon\frac{A^2}{2}\partial_h^2\left\langle v^2 Qf\right\rangle \\ &- \epsilon\frac{A^2}{2}\partial_h\left\langle(1-v^2)Qf\right\rangle + \epsilon\kappa\,\partial_h\left\langle v\cos\phi Qf\right\rangle\end{aligned} \tag{4.16a}$$

and

$$\epsilon\partial_t Qf + v\partial_x Qf - x\partial_v Qf + \omega\partial_\phi Qf - \epsilon\kappa\cos\phi\partial_v Qf - \epsilon\frac{A^2}{2}\partial_v(\partial_v Qf + vQf) + \epsilon\frac{A^2}{4\pi\Omega}\partial_h^2\left\langle v^2\, Qf\right\rangle + \epsilon\frac{\kappa}{2\pi\Omega}\partial_h\left\langle\left(v\cos\phi - \frac{A^2}{2}(1-v^2)\right)Qf\right\rangle = \frac{\kappa}{2\pi}\left[\cos\phi\partial_h\frac{p}{\Omega}\right] + \frac{A^2}{4\pi}\left((v^2-h)\partial_h\frac{p}{\Omega} + (v^2-h)\partial_h^2\frac{p}{\Omega}\right). \tag{4.16b}$$

4.2. First- and second order asymptotic analysis. In this section, we carry on with the above ansatz and show how to derive approximate equations for the energy process. The idea is to express Qf in (4.14) by an approximate solution of (4.15b). We assume that Qf depends on time only through its dependence on p, and use the equation for p in order to calculate $\partial_t Qf$ to the necessary order in ϵ. Once this is substituted into the equation for Qf, we can solve it to the required order and insert the result back into the p-equation. This is exactly the Chapman–Enskog method [5] adapted to this problem. From (4.7) we have the equations

$$\partial_t p - \langle\mathcal{L}_1 Pf\rangle = \epsilon\langle\mathcal{L}_1 Qf\rangle, \tag{4.17a}$$

$$\epsilon\partial_t Qf = \mathcal{L}_2 Qf + \mathcal{L}_1(Pf + \epsilon Qf) - \frac{1}{2\pi\Omega}\langle\mathcal{L}_1(Pf+\epsilon Qf)\rangle. \tag{4.17b}$$

4.2.1. Zeroth order averaging. Solving (4.17) to zeroth order in ϵ yields $\partial_t p - \langle\mathcal{L}_1 Pf\rangle = 0$, which is equivalent to

$$\partial_t p + \frac{A^2}{2}\partial_h(\overline{S(x,v)}p) = \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right). \tag{4.18}$$

This is the same result as is obtained from classical stochastic averaging (3.3).

4.2.2. First order averaging. Solving (4.17) to first order in ϵ leads to the system

$$\partial_t p - \langle\mathcal{L}_1 Pf\rangle = \epsilon\langle\mathcal{L}_1 Qf\rangle, \tag{4.19a}$$

$$-\mathcal{L}_2 Qf = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega} - \frac{1}{2\pi\Omega}\langle\mathcal{L}_1 Pf\rangle. \tag{4.19b}$$

Let us first look at (4.19b) and set $Qf = U_1$. This yields

$$H_v\partial_x U_1 - H_x\partial_v U_1 + c\,\partial_\phi U_1 = \frac{\kappa}{2\pi}\cos\phi\sum_{j=0}^{1}A^j\partial_h^{(j)}\frac{p}{\Omega} + \frac{A^2}{4\pi}\sum_{j=0}^{2}B^j\partial_h^{(j)}\frac{p}{\Omega}, \tag{4.20}$$

with $A^j = A^j(x,v)$ and $B^j = B^j(x,v)$, $\overline{B_j} = 0$. We transform this to a system of simpler equations using the ansatz $U_1 = U_1^\kappa + U_1^A$:

$$U_1^\kappa = \frac{\kappa}{2\pi}\sum_{j=0}^{1}\mathrm{Re}\left\{\left[X^j(x,v)\,\partial_h^{(j)}\left(\frac{p}{\Omega}\right)\right]e^{-i\phi}\right\} = \frac{\kappa}{2\pi}\sum_{j=0}^{1}(M^j\cos\phi + N^j\sin\phi)\partial_h^{(j)}\left(\frac{p}{\Omega}\right), \tag{4.21}$$

$$U_1^A = \frac{A^2}{4\pi}\sum_{j=0}^{2}R^j\partial_h^{(j)}\left(\frac{p}{\Omega}\right), \qquad \overline{R_j} = 0,$$

with $X^j = M^j + iN^j$. This automatically guarantees $\overline{Qf} = 0$. Inserting into the above, we obtain

$$H_v X_x^j - H_x X_v^j - ic\, X^j = A^j \tag{4.22}$$

or

$$H_v M_v^j - H_x M_v^j + c\, N^j = A^j, \tag{4.23a}$$

$$H_v N_v^j - H_x N_v^j - c\, M^j = 0 \tag{4.23b}$$

and

$$H_v R_x^j - H_x R_v^j = B^j, \tag{4.24}$$

and apply the method of characteristics. This gives

$$X^j = X^j(t) = C^j\,\mathrm{Hom}(t) + \mathrm{Inhom}^j(t), \tag{4.25}$$

where C^j are constants,

$$\mathrm{Hom}(t) = \exp\left[i\int_0^t c(x_{t'}, v_{t'})\,dt'\right], \tag{4.26}$$

$$\mathrm{Inhom}^j(t) = \int_0^t A^j(x_{t'}, v_{t'}) \exp\left[i\int_{t'}^t c(x_{t''}, v_{t''})\,dt''\right] dt', \tag{4.27}$$

and

$$R^j = R^j(t) = \int_0^t B^j(x_{t'}, v_{t'})\,dt'. \tag{4.28}$$

Here, (x_t, v_t) is the periodic solution of the unforced deterministic Hamiltonian system. Constants C^j are determined by the required periodicity of X^j, i.e., $X^j(0) \stackrel{!}{=} X^j(T_h)$. However, this is not possible in the case of resonance:

$$\begin{cases} \mathrm{Hom}(T_h) \neq 1 \Rightarrow C = \frac{\mathrm{Inhom}^j(T_h)}{1-\mathrm{Hom}(T_h)}, \\ \mathrm{Hom}(T_h) = 1 \Rightarrow \text{Resonance! No solution.} \end{cases} \tag{4.29}$$

Not surprisingly, the presented ansatz generally fails for the case of resonance. Near the resonance the procedure will not give accurate results. In such a situation there are other approximations of the equations near the resonance that yield more accurate results; see the concluding remarks. In order to detect energy spheres of resonance, we have to find energy levels h such that

$$\int_0^{T_h} c(x_t, v_t)\mathrm{d}t = 2\pi k, \qquad k \in \mathbb{N}. \tag{4.30}$$

Let us henceforth assume that we are not in the resonant case and that periodic $X^j = X^j(t)$ exist. Inserting the approximated solution for Qf into (4.17a), the resulting averaged equation to first order in ϵ then reads

$$\begin{aligned} &\partial_t p + \frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) \\ &= \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_1\right\rangle - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_1\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi U_1\right\rangle \\ &= \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_1^A\right\rangle - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_1^A\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi U_1^\kappa\right\rangle. \end{aligned} \tag{4.31}$$

We note that in general one obtains equations involving fourth order derivatives in h.

Example 1 (cont.). For the linear oscillator, we obtain the first order Q-equation

$$\begin{aligned} &v\partial_x Qf - x\partial_v Qf + \omega\partial_\phi Qf \\ &= \frac{\kappa}{2\pi}\left[v\cos\phi\partial_h\frac{p}{\Omega}\right] + \frac{A^2}{4\pi}\left((v^2-h)\partial_h\frac{p}{\Omega} + (v^2-h)\partial_h^2\frac{p}{\Omega}\right). \end{aligned} \tag{4.32}$$

This leads to $A^1 = v$, $B^1 = B^2 = v^2 - h$. Moreover, $M^1 = \frac{x}{\omega^2-1}$, $N^1 = \frac{-\omega v}{\omega^2-1}$, and $R^1 = R^2 = 2xv$. It is easy to see that all first order coefficients are zero.

REMARK 5. *For the linear oscillator as well as in many other cases, terms of order ϵ are zero, and there is no influence of the periodic forcing on the averaged equations to first order.*

4.2.3. Second order averaging. In order to capture the influence of the forcing on the limit equations we include terms up to order ϵ^2 in system (4.17). We use the ansatz $Qf = U_1 + \epsilon U_2$. Then (4.17b) reads

$$\begin{aligned} \epsilon\partial_t(U_1+\epsilon U_2) &= \mathcal{L}_1 Pf + \mathcal{L}_2(U_1+\epsilon U_2) + \epsilon\mathcal{L}_1(U_1+\epsilon U_2) - \epsilon\left\langle \mathcal{L}_1 U_1 + \epsilon U_2\right\rangle \\ &= \mathcal{L}_1 Pf + \mathcal{L}_2 U_1 + \epsilon\mathcal{L}_2 U_2 + \epsilon\mathcal{L}_1 U_1 + \epsilon^2\mathcal{L}_1 U_2 - \epsilon\left\langle \mathcal{L}_1 U_1\right\rangle - \epsilon^2\left\langle \mathcal{L}_1 U_2\right\rangle. \end{aligned} \tag{4.33}$$

Dropping ϵ^2 terms, we have to solve

$$-\mathcal{L}_2 U_1 = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega} - \frac{1}{2\pi\Omega}\left\langle \mathcal{L}_1 Pf\right\rangle, \qquad (\mathcal{O}(1)) \tag{4.34a}$$

$$-\mathcal{L}_2 U_2 = \mathcal{L}_1 U_1 - \frac{1}{2\pi\Omega}\left\langle \mathcal{L}_1 U_1\right\rangle - \partial_t U_1. \qquad (\mathcal{O}(\epsilon)) \tag{4.34b}$$

Equation (4.34a) has already been solved in the previous section, from which we obtain U_1 as

$$U_1 = \frac{\kappa}{2\pi}\sum_{j=0}^{1}(M^j\cos\phi + N^j\sin\phi)\partial_h^{(j)}\frac{p}{\Omega} + \frac{A^2}{4\pi}\sum_{j=0}^{2}R^j\partial_h^{(j)}\frac{p}{\Omega}. \tag{4.35}$$

We first have to compute $\mathcal{L}_1 U_1$. We rewrite $\mathcal{L}_1$ as

$$\begin{aligned} \mathcal{L}_1 f &= \kappa\cos\phi(\partial_v g f + g\partial_v f) \\ &+ \frac{A^2}{2}\left(f(\partial_v k^2 H_v + k^2\partial_{vv}H) + \partial_v f(\partial_v k^2 + k^2 H_v) + k^2\partial_{vv}f\right). \end{aligned} \tag{4.36}$$

Using the notation $M = (M^0, M^1)$ and in the same way $N = (N^0, N^1)$ and $R = (R^0, R^1, R^2)$, we obtain

$$\begin{aligned} \mathcal{L}_1 U_1 &= \frac{\kappa^2}{2\pi}\sum_{j=0}^{2}\Big(\cos^2\phi G^j(M) + \cos\phi\sin\phi H^j(N)\Big)\partial_h^{(j)}\frac{p}{\Omega} \\ &+ \frac{A^2\kappa}{4\pi}\sum_{j=0}^{3}\Big(\cos\phi(E_1^j(M) + E_2^j(R)) + \sin\phi F^j(N)\Big)\partial_h^{(j)}\left(\frac{p}{\Omega}\right) + \frac{A^4\kappa}{8\pi}\sum_{j=0}^{4}J^j(R)\partial_h^{(j)}\frac{p}{\Omega}, \end{aligned} \tag{4.37}$$

where E_1, E_2, F, G, H, J are suitably defined. Moreover,

$$\frac{1}{2\pi\Omega}\left\langle \mathcal{L}_1 U_1\right\rangle = \frac{\kappa^2}{2\pi}\frac{1}{2}\sum_{j=0}^{2}\left\langle G^j(M)\right\rangle\partial_h^{(j)}\frac{p}{\Omega} + \frac{A^4\kappa}{8\pi}\sum_{j=0}^{4}\left\langle J^j(R)\right\rangle\partial_h^{(j)}\frac{p}{\Omega}. \tag{4.38}$$

Finally, we have to determine $\partial_t U_1$ to zeroth order in ϵ. Using the zeroth order equation for p, one obtains

$$\partial_t p = -\frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) + \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) \tag{4.39}$$

and

$$\partial_t \partial_h^{(j)} p = -\frac{A^2}{2}\,\partial_h^{(j+1)}(\overline{S(x,v)}p) + \frac{A^2}{2}\partial_h^{(j+2)}\left(\overline{H_v^2 k^2}p\right). \tag{4.40}$$

We rewrite $\partial_t U_1$ as

$$\frac{\kappa A^2}{4\pi}\sum_{j=0}^{3}(K^j\cos\phi + L^j\sin\phi)\partial_h^{(j)}\frac{p}{\Omega} + \frac{A^4}{8\pi}\sum_{j=0}^{4}S^j\partial_h^{(j)}\frac{p}{\Omega}, \tag{4.41}$$

with $\langle S^j\rangle = 0$. To solve (4.34b), we split the problem into three equations,

$$\mathcal{L}_2 U_2^\kappa = -(\mathcal{L}_1 U_1)^\kappa + \frac{1}{2\pi\Omega}\,\langle(\mathcal{L}_1 U_1)^\kappa\rangle\,, \tag{4.42a}$$

$$\mathcal{L}_2 U_2^{\kappa A} = -(\mathcal{L}_1 U_1)^{\kappa A} - (\partial_t U_1)^{\kappa A}, \tag{4.42b}$$

$$\mathcal{L}_2 U_2^{A} = -(\mathcal{L}_1 U_1)^{A} + \frac{1}{2\pi\Omega}\,\left\langle(\mathcal{L}_1 U_1)^{A}\right\rangle - (\partial_t U_1)^{A}\,, \tag{4.42c}$$

where the right-hand side in the first equation includes the parts of order κ^2 in (4.17b) and the right-hand side in the second equation includes those of order κA^2, whereas the third equation deals with terms of order A^4. System (4.42) is solved in the same manner as the first order approximation: The first equation, (4.42a), is solved with the ansatz

$$U_2^\kappa = \kappa^2\frac{1}{2\pi}\sum_{j=0}^{2}\left((M^\kappa)^j\left(\cos^2\phi - \frac{1}{2}\right) + (N^\kappa)^j\cos\phi\sin\phi + (R^\kappa)^j\right)\partial_h^{(j)}\frac{p}{\Omega}, \tag{4.43}$$

where $\langle R^\kappa\rangle = 0$ is required to obtain $\overline{U_2^\kappa} = 0$, which is a condition for fixing R^κ. As before, we require periodicity of M^κ and N^κ. The second equation, (4.42b), is solved by introducing

$$U_2^{\kappa A} = \frac{A^2\kappa}{4\pi}\sum_{j=0}^{3}\left((M^{\kappa A})^j\cos\phi + (N^{\kappa A})^j\sin\phi\right)\partial_h^{(j)}\frac{p}{\Omega}. \tag{4.44}$$

This leads to similar equations as in the first order case. Note that $\langle U_2^{\kappa A}\rangle = 0$ is automatically fulfilled. Applying the method of characteristics gives the solution $X^{\kappa A} = M^{\kappa A} + iN^{\kappa A}$. Again periodicity of $X^{\kappa A}$ is required. Finally, the third equation, (4.42c), is solved with the ansatz

$$U_2^A = \frac{A^4}{4}\frac{1}{2\pi}\sum_{j=0}^{4}(R^A)^j\partial_h^{(j)}\frac{p}{\Omega} \tag{4.45}$$

and again the requirement $\langle R^A \rangle = 0$. Putting all results together, this yields the final second order contributions to (4.15a), which now read

$$\begin{aligned}
&\partial_t p + \frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) \\
&= \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_1^A\right\rangle - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_1^A\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi U_1^\kappa\right\rangle \\
&\quad + \epsilon^2\kappa\,\partial_h\left\langle H_v g\cos\phi U_2^{\kappa A}\right\rangle + \epsilon^2\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2(U_2^\kappa + U_2^A)\right\rangle \\
&\quad - \epsilon^2\frac{A^2}{2}\,\partial_h\left\langle S(x,v)(U_2^\kappa + U_2^A)\right\rangle.
\end{aligned} \tag{4.46}$$

We note that the resulting equations involve sixth order derivatives in h and averages of the functions $M^{\kappa A}$ and R^κ, R^A. In this way one obtains averaged equations where the coefficients are again computed by time averages over the unforced Hamiltonian motion of the original system. However, the appearing functions and their averages are in general not explicitly computable. We have to use numerical methods to determine the stochastic coefficients. In the next section, we present a simplified averaging procedure which will then be used together with a numerical approach to compare the stationary distribution of the full system (2.2) and the higher order approximate energy process, as well as the relaxation of both processes to equilibrium.

4.3. Further simplifications. In this section, we derive further approximations using an additional assumption to simplify the Q-equation. This assumption will yield accurate results as long as we are in a near-equilibrium situation. We make the following "near-equilibrium" assumption, which will produce simpler limit equations.

ASSUMPTION 1. *Proximity to equilibrium in p: Approximate* $\partial_h \frac{p}{\Omega} = \partial_h P f$ *by* $-\frac{p}{\Omega} = -Pf$ *in* (4.15b).

REMARK 6. *Using this assumption, we will neglect several terms compared to the formal second order asymptotic analysis from section* 4.2. *However, in a near-equilibrium situation they are expected to be small. These statements are supported by the numerical results in the following section.*

With this assumption we have $\langle \mathcal{L}_1 P f\rangle = 0$, and the Q-equation becomes

$$\epsilon\partial_t Qf - \mathcal{L}_2 Qf - \epsilon\mathcal{L}_1 Qf = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega} - \epsilon\frac{1}{2\pi\Omega}\left\langle \mathcal{L}_1 Qf\right\rangle. \tag{4.47}$$

Note that to leading order this yields that $-\mathcal{L}_2 Qf$ is equal to a functional of p. Thus we can find a leading order solution U_1 which depends on p and satisfies $\langle Qf\rangle = 0$. Using now the near equilibrium assumption in (4.39) yields $\partial_t p = 0$ to zeroth order. Moreover, this gives to leading order (see (4.41)) that $\partial_t Qf = 0$. Thus, in the Q-equation, we can neglect $\epsilon\partial_t Qf$ as long as we do not proceed beyond a second order analysis. We consider

$$\partial_t p - \left\langle \mathcal{L}_1 P f\right\rangle = \epsilon\left\langle \mathcal{L}_1 Qf\right\rangle, \tag{4.48a}$$

$$-\mathcal{L}_2 Qf - \epsilon\mathcal{L}_1 Qf = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega} - \epsilon\frac{1}{2\pi\Omega}\left\langle \mathcal{L}_1 Qf\right\rangle. \tag{4.48b}$$

Explicitly, we obtain

$$\begin{aligned}
&\partial_t p + \frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right) = +\epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 Qf\right\rangle \\
&\quad - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)Qf\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi Qf\right\rangle
\end{aligned} \tag{4.49a}$$

and

$$
\begin{aligned}
&H_v\partial_x Qf - H_x\partial_v Qf + \partial_\phi(c\,Qf)\\
&-\epsilon\kappa g\cos\phi\partial_v Qf - \epsilon\frac{A^2}{2}\partial_v(k^2(\partial_v Qf + Qf\partial_v H))\\
&+\epsilon\frac{A^2}{4\pi\Omega}\partial_h^2\left\langle H_v^2k^2\,Qf\right\rangle + \epsilon\frac{\kappa}{2\pi\Omega}\partial_h\left\langle\left(H_v g\cos\phi - \frac{A^2}{2}S(x,v)\right)Qf\right\rangle\\
&=\frac{\kappa}{2\pi}\left((g_v - H_v g)\cos\phi\frac{p}{\Omega}\right).
\end{aligned}
\tag{4.49b}
$$

In the following, we solve systems (4.48), (4.49) up to first and second order in ϵ using the above assumption. We note that obviously zeroth order averaging yields the same result as in the last section.

4.3.1. First order averaging. Solving (4.48) to first order in ϵ leads to the system

$$\partial_t p - \langle\mathcal{L}_1 Pf\rangle = \epsilon\langle\mathcal{L}_1 Qf\rangle, \tag{4.50a}$$

$$-\mathcal{L}_2 Qf = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega}. \tag{4.50b}$$

Let us first look at (4.50b) and set $Qf = U_1$. This yields

$$H_v\partial_x U_1 - H_x\partial_v U_1 + c\,\partial_\phi U_1 = \frac{\kappa}{2\pi}\cos\phi\,(g_v - H_v g)\,\frac{p}{\Omega}, \tag{4.51}$$

which we transform to a system of simpler equations using the ansatz

$$U_1 = \kappa\mathrm{Re}\left\{\left[X(x,v)\,\frac{p}{2\pi\Omega}\right]e^{-i\phi}\right\} = \kappa\frac{p}{2\pi\Omega}(M\cos\phi + N\sin\phi), \tag{4.52}$$

with $X = M + iN$. This automatically guarantees $\overline{Qf} = 0$. Inserting into the above, we obtain

$$H_v X_x - H_x X_v - ic\,X = g_v - H_v g \tag{4.53a}$$

or

$$H_v M_v - H_x M_v + c\,N = g_v - H_v g, \tag{4.54a}$$

$$H_v N_v - H_x N_v - c\,M = 0, \tag{4.54b}$$

and apply the method of characteristics as in section 4.2. Again

$$X = X(t) = C\,\mathrm{Hom}(t) + \mathrm{Inhom}(t), \tag{4.55a}$$

where

$$\mathrm{Hom}(t) = \exp\left[i\int_0^t c(x_{t'}, v_{t'})\,dt'\right], \tag{4.55b}$$

$$\mathrm{Inhom}(t) = \int_0^t (g_v - H_v g)(t')\exp\left[i\int_{t'}^t c(x_{t''}, v_{t''})\,dt''\right]dt', \tag{4.55c}$$

and C is determined by the periodicity of X; i.e., $C = \frac{\mathrm{Inhom}(T_h)}{1-\mathrm{Hom}(T_h)}$ in the nonresonant case. Inserting the approximated solution for Qf into (4.50a), the resulting averaged equation to first order then reads

$$\begin{aligned}
&\partial_t p + \frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right)\\
&= \epsilon\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_1\right\rangle - \epsilon\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_1\right\rangle + \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi U_1\right\rangle\\
&= \epsilon\kappa\,\partial_h\left\langle H_v g\cos\phi U_1\right\rangle = \epsilon\frac{\kappa^2}{2}\,\partial_h(\overline{H_v g M}p).
\end{aligned} \tag{4.56}$$

The corresponding averaged stochastic coefficients are

$$a(h) = \frac{A^2}{2}\overline{S(x,v)} - \frac{\epsilon\kappa^2}{2}\overline{H_v g M}, \tag{4.57}$$

$$\sigma^2(h) = A^2\partial_h^2\overline{H_v^2 k^2}. \tag{4.58}$$

REMARK 7. *As noticed before for the linear oscillator, these terms of order ϵ are zero and there is no influence of the periodic forcing on the averaged equations to first order.*

4.3.2. Second order averaging. In order to capture the influence of the forcing on the limit equations we include terms up to order ϵ^2 in system (4.48). We use the ansatz $Qf = U_1 + \epsilon U_2$. Then (4.7b) reads

$$\begin{aligned}
0 &= \mathcal{L}_1 Pf + \mathcal{L}_2(U_1+\epsilon U_2) + \epsilon\mathcal{L}_1(U_1+\epsilon U_2) - \epsilon\left\langle \mathcal{L}_1 U_1 + \epsilon U_2\right\rangle\\
&= \mathcal{L}_1 Pf + \mathcal{L}_2 U_1 + \epsilon\mathcal{L}_2 U_2 + \epsilon\mathcal{L}_1 U_1 + \epsilon^2\mathcal{L}_1 U_2 - \epsilon\left\langle \mathcal{L}_1 U_1\right\rangle - \epsilon^2\left\langle \mathcal{L}_1 U_2\right\rangle.
\end{aligned} \tag{4.59}$$

Dropping $\epsilon^2\mathcal{L}_1 U_2$, we have to solve

$$-\mathcal{L}_2 U_1 = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega}, \qquad (\mathcal{O}(1)) \tag{4.60a}$$

$$-\mathcal{L}_2 U_2 = \mathcal{L}_1 U_1 - \left\langle \mathcal{L}_1 U_1\right\rangle. \qquad (\mathcal{O}(\epsilon)) \tag{4.60b}$$

Equation (4.60a) has already been solved in the previous section, from which we obtain U_1. We first have to compute $\mathcal{L}_1 U_1$. We rewrite $\mathcal{L}_1$ as

$$\begin{aligned}
\mathcal{L}_1 f &= \kappa\cos\phi(\partial_v g f + g\partial_v f)\\
&+ \frac{A^2}{2}\left(f(\partial_v k^2 H_v + k^2\partial_{vv}H) + \partial_v f(\partial_v k^2 + k^2 H_v) + k^2\partial_{vv}f\right).
\end{aligned} \tag{4.61}$$

Then, using $U_1 = \kappa(M\cos\phi + N\sin\phi)\frac{p}{2\pi\Omega}$ and inserting Assumption 1, we obtain

$$\begin{aligned}
\mathcal{L}_1 U_1 = \frac{p}{2\pi\Omega}\Big(&\kappa^2\cos^2\phi G(M) + \kappa^2\cos\phi\sin\phi H(N)\\
&+ \frac{A^2\kappa}{2}\cos\phi F(M) + \frac{A^2\kappa}{2}\sin\phi F(N)\Big),
\end{aligned} \tag{4.62}$$

where G, H, J, F are defined as $G(M) = M(g_v - H_v g) + gM_v$, $H(N) = N(g_v - gH_v) + gN_v$, $F(M) = M_v(k_v^2 - k^2 H_v) + k^2 M_{vv}$. Moreover,

$$\left\langle \mathcal{L}_1 U_1\right\rangle = \frac{p}{2\pi\Omega}\kappa^2\frac{1}{2}\left\langle G(M)\right\rangle. \tag{4.63}$$

To solve (4.60b), we split the problem into two equations,

$$\mathcal{L}_2 U_2^\kappa = -(\mathcal{L}_1 U_1)^\kappa + \langle \mathcal{L}_1 U_1 \rangle \,, \tag{4.64}$$

$$\mathcal{L}_2 U_2^{\kappa A} = -(\mathcal{L}_1 U_1)^{\kappa A}, \tag{4.65}$$

where the right-hand side in the first equation includes the parts of order κ^2 in (4.62) and the right-hand side in the second equation includes those of order κA^2. Equation (4.65) is solved in the same way as the first order approximation or as the corresponding equation in section 4.2. We introduce

$$U_2^{\kappa A} = \frac{A^2}{2}\kappa(M^{\kappa A}\cos\phi + N^{\kappa A}\sin\phi)\frac{p}{2\pi\Omega}. \tag{4.66}$$

This yields, as for first order,

$$H_v M_x^{\kappa A} - H_x M_v^{\kappa A} + c\, N^{\kappa A} = F(M), \tag{4.67a}$$

$$H_v N_x^{\kappa A} - H_x N_v^{\kappa A} - c\, M^{\kappa A} = F(N). \tag{4.67b}$$

Applying the method of characteristics gives the solution $X^{\kappa A} = M^{\kappa A} + iN^{\kappa A}$. Again periodicity of $X^{\kappa A}$ is required. The resulting averages in (4.15a) are

$$\begin{aligned} \epsilon^2\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_2^{\kappa A}\right\rangle - \epsilon^2\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_2^{\kappa A}\right\rangle + \epsilon^2\kappa\,\partial_h\left\langle H_v g\cos\phi U_2^{\kappa A}\right\rangle \\ = \epsilon^2\frac{A^2\kappa^2}{4}\partial_h(\overline{H_v g M^{\kappa A}}p). \end{aligned} \tag{4.68}$$

Finally we have to solve (4.64). The ansatz for U_2^κ is again, as in section 4.2, given by

$$U_2^\kappa = \kappa^2\frac{p}{2\pi\Omega}\left(M^\kappa\left(\cos^2\phi - \frac{1}{2}\right) + N^\kappa\cos\phi\sin\phi + R^\kappa\right). \tag{4.69}$$

This gives

$$\dot{X^\kappa} = i2c\,X + G(M) + iH(N), \tag{4.70a}$$

$$\dot{R^\kappa} = \frac{1}{2}(G(M) - \langle G(M)\rangle), \tag{4.70b}$$

with $X^\kappa = M^\kappa + iN^\kappa$. As before, we require periodicity of X^κ. Moreover, in this case, $\overline{U_2^\kappa}$ has to be equal to 0 to get $\overline{Qf} = 0$. This is true if $\overline{R^\kappa} = 0$, which is a condition for fixing R^κ. This yields

$$R^\kappa = \int_0^t \frac{1}{2}(G(M) - \langle G(M)\rangle)\,\mathrm{d}t'. \tag{4.71}$$

In (4.15a), we obtain the following averages:

$$\begin{aligned} \epsilon^2\frac{A^2}{2}\partial_h^2\left\langle H_v^2 k^2 U_2^{\kappa}\right\rangle - \epsilon^2\frac{A^2}{2}\,\partial_h\left\langle S(x,v)U_2^{\kappa}\right\rangle + \epsilon^2\kappa\,\partial_h\left\langle H_v g\cos\phi U_2^{\kappa}\right\rangle \\ = \epsilon^2\frac{A^2\kappa^2}{2}\partial_{hh}(\overline{k^2 H_v^2 R^\kappa}p) - \epsilon^2\frac{A^2\kappa^2}{2}\partial_h(\overline{S R^\kappa}p). \end{aligned} \tag{4.72}$$

Putting all the results together yields the final equation for the approximated energy dynamics:

$$\begin{aligned}&\partial_t p + \frac{A^2}{2}\,\partial_h(\overline{S(x,v)}p) - \frac{A^2}{2}\partial_h^2\left(\overline{H_v^2 k^2}p\right)\\ &= \epsilon\frac{\kappa^2}{2}\,\partial_h(\overline{H_v g M}p) + \epsilon^2\frac{A^2\kappa^2}{4}\partial_h(\overline{H_v g M^A}p) - \epsilon^2\frac{A^2\kappa^2}{2}\partial_h(\overline{SR^\kappa}p)\\ &\quad + \epsilon^2\frac{A^2\kappa^2}{2}\partial_{hh}(\overline{k^2 H_v^2 R^\kappa}p).\end{aligned} \tag{4.73}$$

The corresponding averaged stochastic coefficients in second order are

$$a(h) = \frac{A^2}{2}\overline{S(x,v)} - \epsilon\frac{\kappa^2}{2}\overline{H_v g M} - \epsilon^2\frac{A^2\kappa^2}{4}(\overline{H_v g M^{\kappa A}}) + \epsilon^2\frac{A^2\kappa^2}{2}(\overline{SR^\kappa}), \tag{4.74a}$$

$$\sigma^2(h) = A^2\overline{H_v^2 k^2} + \epsilon^2 A^2\kappa^2(\overline{k^2 H_v^2 R^\kappa}). \tag{4.74b}$$

REMARK 8. *We note that compared to the asymptotic analysis from section* 4.2 *where the near equilibrium assumption has not been used, there are several terms approximated by zero. In particular, this is the case for terms involving* U_2^A *in* (4.46). *These terms do not cancel out in a full asymptotic analysis. However, in a near equilibrium situation they are expected to be small. These statements are supported by the numerical results in the next section.*

We have obtained averaged stochastic coefficients of higher order, which are again computed by time averages over the unforced Hamiltonian motion of the original system. However, the functions that appear and their averages are in general not explicitly computable. We have to use numerical methods to determine the stochastic coefficients. In the next section, we present our numerical approach together with numerical results for the comparison of the stationary distribution of the full system (2.2) and the higher order approximate energy process (4.73) as well as the relaxation of both processes to equilibrium.

Example 1 (cont.). For the linear oscillator, we obtain explicit results for the nonresonant case $\omega \neq 1$:

$$M^{\kappa A} = -\frac{\omega^2}{(\omega^2-1)^2}v, \qquad N^{\kappa A} = -\frac{\omega}{(\omega^2-1)^2}x, \tag{4.75}$$

$$M^\kappa = \frac{\omega^2+1}{4(\omega^2-1)^2}\left(x^2-v^2\right), \quad N^\kappa = -\frac{\omega}{(\omega^2-1)^2}xv, \quad R^\kappa = \frac{1}{4(\omega^2-1)}(v^2-h).$$

The final coefficients are

$$a(h) = \frac{A^2}{2}(1-h) + \frac{\epsilon^2A^2\kappa^2}{4}\left(\frac{\omega^2}{(\omega^2-1)^2}h - \frac{1}{4(\omega^2-1)}h^2\right), \tag{4.76a}$$

$$\sigma^2(h) = A^2 h + \frac{\epsilon^2A^2\kappa^2}{2}\frac{1}{4(\omega^2-1)}h^2. \tag{4.76b}$$

We note that the above formulas have a restricted range of validity. The resulting equations are meaningful only as long as $\sigma^2(h)$ is nonnegative. This yields for $\omega < 1$ an upper bound for the energy variable; see the next section. We mention that these explicit formulas are used as a benchmark for the numerical computations in section 5.1. See Appendix A for details.

REMARK 9. *In the linear case with* $\omega = 0$*, i.e.,* $\phi = \text{const.}$*, a straightforward argument connects the stationary distributions of the full and reduced problems. Expanding the equilibrium of the full Fokker–Planck equation* (2.4) *in* ϵ *around* $\epsilon = 0$ *and averaging* $x^2 \to \overline{x^2} = h$*,* $\cos\phi \to \overline{\cos\phi} = 0$*,* $\cos^2\phi \to \overline{\cos^2\phi} = \frac{1}{2}$*, we obtain*

$$p_s(h) \approx e^{-h}\left(1 + \frac{\epsilon^2\kappa^2}{4}h\right) \tag{4.77}$$

in second order in ϵ*. No terms of first order in* ϵ *appear. The same result is obtained if we compute for* $\omega = 0$ *the stationary distribution of the one-dimensional Fokker–Planck equation with coefficients* (4.76a) *and expand the result again in* ϵ *around* $\epsilon = 0$*. See Appendix* A *for more details.*

5. Numerical results. In this section the numerical methodology for determining the numerical averages for the limiting coefficients is explained in detail, and the two examples described above are investigated numerically.

5.1. Numerical methodology. For classic stochastic averaging, finding drift and diffusion coefficients is reduced to averaging along Hamiltonian spheres,

$$\bar{f}(h) := \frac{1}{T_h}\int_0^{T_h} f(x_t, v_t)\mathrm{d}t, \tag{5.1}$$

which can be treated easily with standard numerical methods. The evaluation of averages is also a numerical component of the presented higher order ansatz (4.73). Additionally, further integrals along spheres have to be evaluated involving M (see (4.57)) for first order, and $M^{\kappa A}$ and R^{κ} (see (4.74)) for second order averaging. In our code, we use a standard adaptive fourth order Runge–Kutta method to solve the deterministic Hamiltonian system (2.2) (with $\kappa = 0$, $A = 0$) for a fixed number of energy values. Note that the time grid $[0, T_h]$ is sufficient, but the period length T_h is a priori unknown. Numerical integration along the adaptive time grid is done with a standard second order quadrature formula.

A more challenging problem is the numerical spatial differentiation of M, N in the second order procedure. Recall that M_v, M_{vv}, N_v form the right-hand sides of (4.67), (4.70). We have to compute a partial derivative in space, but function values are given on the adaptively discretized Hamiltonian spheres. Instead of a regular grid, data is available only on the point cloud illustrated in Figure 5.1. To overcome this difficulty, we use a moving least squares approximation (see [11]), which is similar to the differentiation techniques used in mesh-free methods (see, e.g., [14]). Fix any point $p = (x, y) \in \mathbb{R}^2$, and let $B_{p,r}$ be the set of neighboring data points contained in a ball of radius r. By a Taylor expansion for any $p_i = (x_i, y_i) \in B_{p,r}$

$$f(p_i) = f(p) + \nabla f(p)\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\Delta x & \Delta y\end{pmatrix}\cdot \text{Hess } f(p)\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} + e_i, \tag{5.2}$$

where $\Delta x = x_i - x$, $\Delta y = y_i - y$, and e_i denotes the error term. Writing the above for all points z_i and introducing an error weighting function $w = w(\|z_i - z\|)$, the weighted sum of all e_i is a quadratic form J of the five unknowns $\partial_x f(p)$, $\partial_y f(p)$, $\partial_{x^2} f(p)$, $\partial_x\partial_y f(p)$, $\partial_{y^2} f(p)$. The minimizing condition $\nabla J = 0$ leads to a linear system to be solved. This procedure has to be repeated for all data points of the discretized Hamiltonian spheres.

In contrast to some mesh-free methods, the data points are fixed by the adaptive ODE solver. We therefore have to filter points in the neighborhood $B_{z,r}$ to avoid an

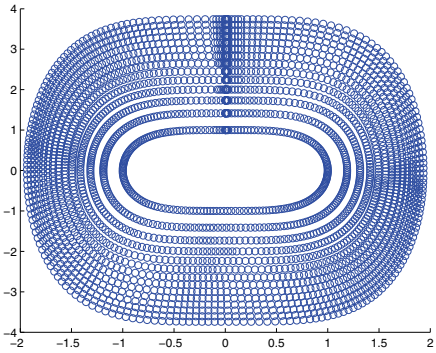


FIG. 5.1. *Irregular grid from discretized Hamiltonian spheres for the nonlinear oscillator (*14 *spheres,* $H \in [\frac{1}{2}, 7]$*).*

ill-conditioned linear system. This can be achieved by (1) limiting the total number of neighbors considered, (2) avoiding extremely close neighbors, and (3) avoiding multiple points on the same line from the central point z. The last filter avoids linear dependence of rows in the linear system. In our code, we divide neighbors into six equal angular segments and accept up to five closest points in each segment.

5.2. The case of a linear potential. We compare the numerical solution of the full energy process

$$\mathrm{d}H_t^\epsilon = -\kappa v \cos\phi \mathrm{d}t + \frac{A^2}{2}(1-v^2)\mathrm{d}t + Av\mathrm{d}W_t, \tag{5.3}$$

where x, v, ϕ solve (2.2) with the averaging result, i.e., the reduced equation

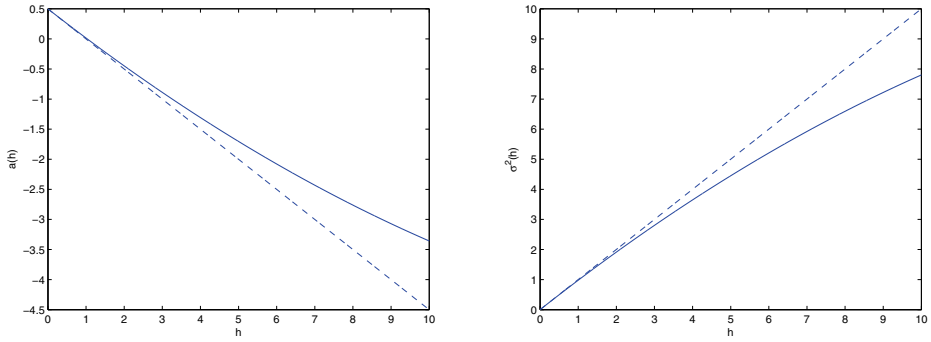
$$\mathrm{d}H_t^{av} = a(H_t^{av})\mathrm{d}t + \sigma(H_t^{av})\,\mathrm{d}W_t \tag{5.4}$$

with drift and variance

$$a(h) = \frac{A^2}{2}(1-h) + \frac{\epsilon^2 A^2 \kappa^2}{4}\left(\frac{\omega^2}{(\omega^2-1)^2}h - \frac{1}{4(\omega^2-1)}h^2\right), \tag{5.5a}$$

$$\sigma^2(h) = A^2 h + \frac{\epsilon^2 A^2 \kappa^2}{2}\frac{1}{4(\omega^2-1)}h^2, \tag{5.5b}$$

with $\omega > 1$ or $\omega < 1$ and $h < \frac{8(1-\omega^2)}{\epsilon^2\kappa^2}$ to guarantee positivity of $\sigma^2(h)$. First we look at the stationary energy distribution, second we compare the relaxation into equilibrium. Changes in the considered observables, say stationary distribution, induced by periodic forcing can be very small. To illustrate the ratio of higher order terms in the averaged coefficients with previous zeroth order results, we first plot the drift and diffusion coefficients for the case $\kappa = 0.4$, $\omega = 0.3$, $A = 1$, $\epsilon = 1$ in Figure 5.2. The stationary distributions are investigated for various values of the parameters κ, ω. We compare the stationary solution of the oscillator without forcing, $p_{s,\,\kappa=0}(h) = e^{-h}$, with the stationary distribution obtained from a Monte Carlo simulation of the full system ((2.4), (2.5), (5.3)) and the stationary distribution resulting from higher order averaging ((5.4), (5.5a)) in Figure 5.3: Monte Carlo simulations were done with $N = 20000$ runs and initial values uniformly distributed in ϕ and δ-distributed in



(a) $a(h)$. (b) $\sigma^2(h)$.

FIG. 5.2. *Drift and diffusion coefficients for the linear oscillator for parameters* $\kappa = 0.4$, $\omega = 0.3$, $A = 1$ *(solid −) and unforced case* $\kappa = 0$, $A = 1$ *(dashed - -).*

(x, v) at zero. The time step was $\Delta t = 10^{-3}$ with time interval $[0, 1000]$. The asymptotic stationary distribution of higher order averaging is computed based on the higher order coefficients via

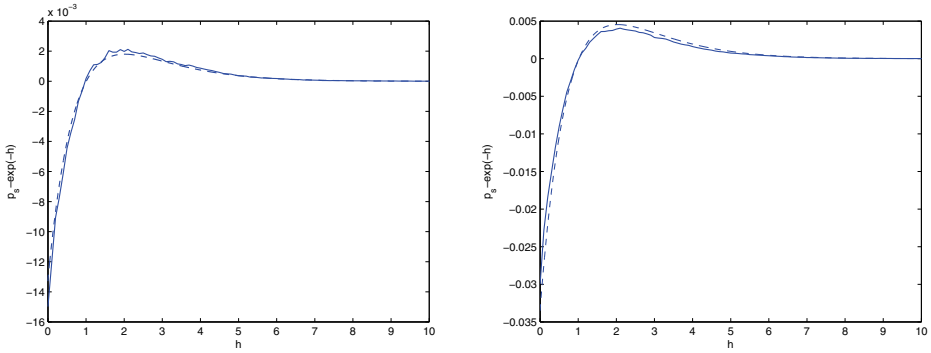
$$p_s(h) = C\exp\left(\int_0^h \frac{2a - \partial_h\sigma^2}{\sigma^2}dh'\right). \tag{5.6}$$

We remark that, for $\omega < 1$, there exists an energy level $h^* = \frac{8(1-\omega^2)}{\epsilon^2\kappa^2}$ such that $\sigma^2(h^*) = 0$. If h^* is within the range of relevant energy levels, the asymptotic stationary distribution breaks down. For κ large and for values of ω near 1 the approximation ceases to be valid. Moreover, for $\omega = 1$, the approximation breaks down due to resonance, which can easily be seen from condition (4.30). However, in the cases presented here h^* is very large ($h^* \gg 10$), representing extremely unlikely energy levels. The results in Figure 5.3 show that higher order averaging works well in predicting stationary distributions if the system is away from resonance. The slight changes in distribution induced by periodic forcing are reproduced at much lower computational cost and a better understanding of the underlying energy dynamics. As the forcing parameter increases, the predicted pertubation is qualitatively correct and a definite improvement over zeroth order averaging. For parameters near resonance, the approximation will fail.

In Figure 5.4, we compare the relaxation into equilibrium for the full system ((2.4), (2.5), (5.3)) and the reduced equation ((5.4), (5.5a)) based on higher order coefficients: The initial energy distribution is uniformly distributed in $[0, 5]$, and the Monte Carlo simulations both included $N = 500000$ paths, with $\Delta t = 10^{-3}$ on the time interval $[0, 40]$. The distance of the energy distribution to its equilibrium at different times is measured by the relative entropy

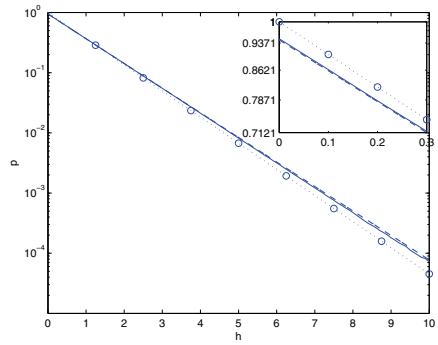
$$D(p, p_s) := \int p(h) \cdot \log\left(\frac{p(h)}{p_s(h)}\right) \mathrm{d}h. \tag{5.7}$$

We observe that the relaxation of the energy process of the full system is adequately reproduced by the reduced equation. The higher order averaging procedure allows a better understanding not only of equilibria but also of the underlying dynamics.

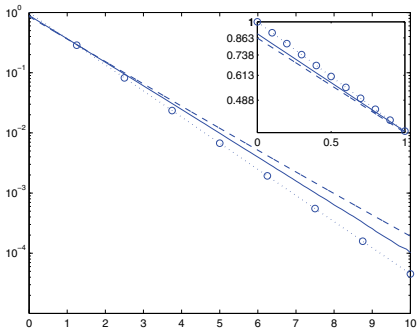


(a) $A = 1$, $\kappa = 0.2$, $\omega = 0.3$.

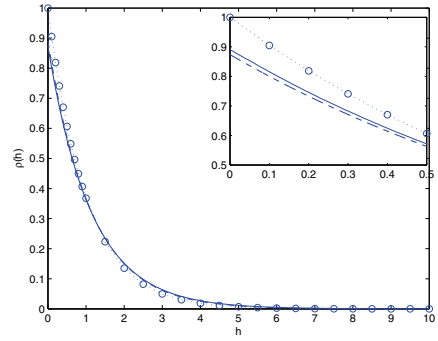
(b) $A = 1$, $\kappa = 0.2$, $\omega = 0.6$.



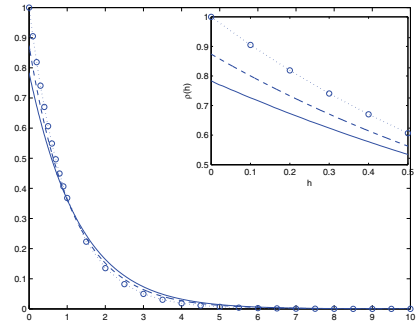
(c) $A = 1$, $\kappa = 0.4$, $\omega = 0.3$.



(d) $A = 1$, $\kappa = 0.4$, $\omega = 0.6$.

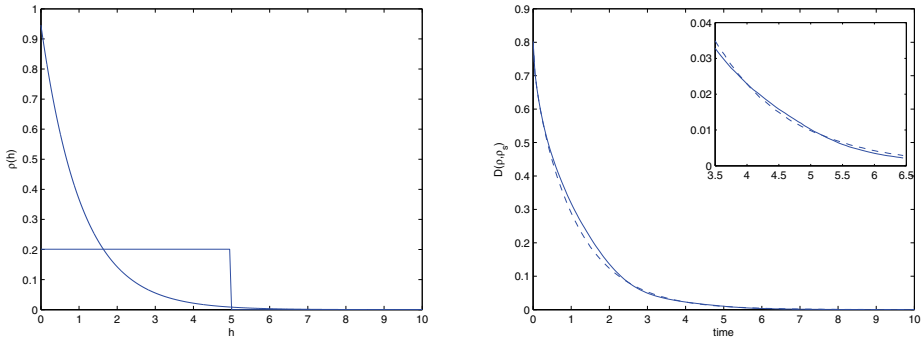


(e) $A = 1$, $\kappa = 0.6$, $\omega = 0.3$.



(f) $A = 1$, $\kappa = 0.6$, $\omega = 0.6$.

FIG. 5.3. *Stationary distribution of linear oscillator with different values of* κ *and* ω*:* $(\cdots\circ)$ *unforced stationary solution,* $(-)$ *full SDE system by Monte Carlo simulation,* $(--)$ *asymptotic results of higher order averaging. In* (a) *and* (b) *we plot the difference to the unforced stationary distribution* e^{-h}*, in* (c) *and* (d) *we plot the distributions on a logarithmic scale, and in* (e) *and* (f) *on a regular scale. For all cases* $A = 1$, $\epsilon = 1$*. The forcing strength varies by line in* $\kappa \in \{0.2\,, 0.4\,, 0.6\}$*; the forcing frequency varies by column in* $\omega \in \{0.3\,, 0.6\}$.



(a) Initial and stationary distributions.

(b) Time evolution of relative entropy D: (-) full system, (- -) reduced equation.

FIG. 5.4. *Relaxation into equilibrium for the linear oscillator with* $\kappa = 0.4$, $\omega = 0.3$. *The energy dynamics of the full system is adequately approximated by the reduced energy dynamics with higher order coefficients.*

5.3. The case of a nonlinear potential. In our second example we choose the nonlinear potential $V(x) = \frac{x^{2n}}{2n}$ with $n = 2$,

$$\partial_t f + v\partial_x f - x^{2n-1}\partial_v f - \kappa\partial_v(\cos\phi f) + \partial_\phi(cf) = \frac{A^2}{2}\partial_v(\partial_v f + vf). \tag{5.8}$$

In this case the solutions of the unforced deterministic system with energy h can be written as

$$x(t) = h^{\frac{1}{2n}} x_1(\varphi(t)), \tag{5.9a}$$

$$v(t) = h^{\frac{1}{2}} v_1(\varphi(t)), \tag{5.9b}$$

where $\varphi(t) = \varphi_0 + h^{\frac{1}{2}-\frac{1}{2n}}t$ and x_1 is the solution for energy $h = 1$. φ_0 is determined by the initial values. The period of motion is

$$T_h = \frac{T_1}{h^{\frac{1}{2}-\frac{1}{2n}}}, \tag{5.10}$$

and the virial theorem states $\overline{x^{2n}} = \frac{2n}{1+n}h = \overline{v^2}$. This yields the zeroth order Fokker–Planck equation

$$\partial_t p + \partial_h(a(h)p) = \frac{1}{2}\partial_{hh}(\sigma^2(h)p), \tag{5.11}$$

with

$$a(h) = \frac{A^2}{2}\overline{1-v^2} - \kappa\overline{v\cos\phi} = \frac{A^2}{2}\left(1 - \frac{2n}{n+1}h\right), \tag{5.12a}$$

$$\sigma^2(g) = A^2\frac{2n}{n+1}h. \tag{5.12b}$$

The resulting stationary distribution, which is equal to the stationary energy distribution of the unforced full system ($\kappa = 0$), is $p_s = Ce^{-h}T_h = C\frac{e^{-h}}{h^{\frac{1}{2}-\frac{1}{2n}}}$. If we choose

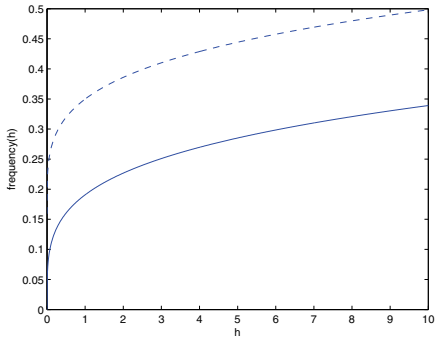


FIG. 5.5. *Nonlinear example: Hamiltonian frequency* $\frac{1}{T_h}$ *(– solid) and energy dependent periodic force* $c(h)$ *with* $d = 1$ *(- - dashed).*

the periodic forcing as $c(x, v) := \omega$, as in Example 1, we will face resonance, because there will exist an h such that $\frac{2\pi}{\omega} = T_h$. We discuss this case in Remark 11 below and choose instead

$$c(x,v) := \frac{2\pi}{T_{H(x,v)}} + d = \frac{2\pi}{T_1}(H(x,v))^{\frac{1}{4}} + d, \qquad d > 0, \tag{5.13}$$

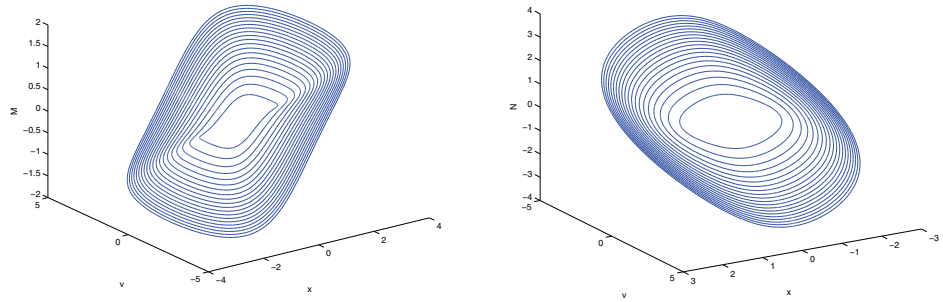
in order to avoid resonance; see Figure 5.5. In this case the first- and second order coefficients have to be computed numerically using the methods explained in section 5.1 and the results in section 4. As in Example 1, first order coefficients are zero. The averaged SDE is $\mathrm{d}H_t^{av} = a(H_t^{av})\mathrm{d}t + \sigma(H_t^{av})\,\mathrm{d}W_t$ with drift and variance

$$a(h) = \frac{A^2}{2}\left(1 - \frac{2n}{n+1}h\right) - \epsilon^2\frac{A^2\kappa^2}{4}\overline{vM^{\kappa A}} - \epsilon^2\frac{A^2\kappa^2}{2}\overline{v^2R^{\kappa}}, \tag{5.14a}$$

$$\sigma^2(h) = A^2\frac{2n}{n+1}h + \epsilon^2A^2\kappa^2\overline{v^2R^{\kappa}}. \tag{5.14b}$$

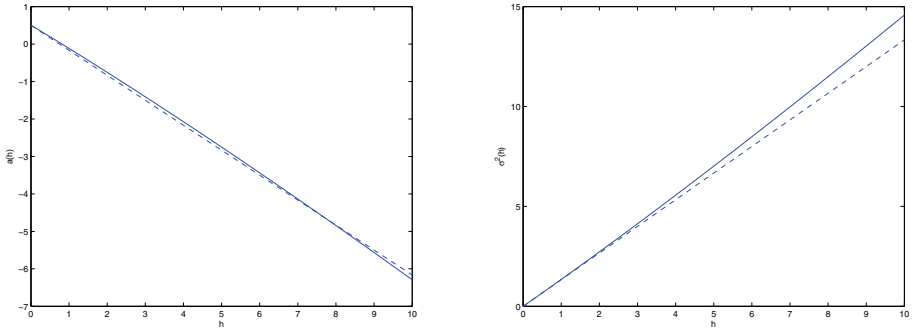
The diffusion coefficient is positive as long as either $\overline{v^2R^{\kappa}}$ is positive or $\overline{v^2R^{\kappa}}$ is negative and $h < -\frac{2n}{(n+1)\epsilon^2\kappa^2\overline{v^2R^{\kappa}}}$. The stationary distribution is computed via (5.6). We illustrate the numerical computation by showing the surfaces M and N from (4.54) for a set of energy spheres in Figure 5.6. In Figure 5.7 the resulting drift and diffusion coefficients (5.14) are plotted for the example $\kappa = 0.6$, $d = 1$. As before, we investigate stationary energy distributions of the full system (5.8) and the reduced equation with the unforced equilibrium. Monte Carlo simulations of the full system were done with $N = 50000$ runs and initial values uniformly distributed in ϕ and the energy of (x, v)-space uniformly distributed in $[0, 5]$. The time step was $\Delta t = 10^{-3}$ with time interval $[0, 80]$. The results are shown in Figure 5.8. We see that for periodic forcing with some distance d to the resonance, the higher order averaging procedure predicts very well the influence of the forcing. Small changes, which are expensive to obtain from MC-simulations, are reproduced accurately. For $d = 0.5$, results do not match as nicely but give a good implication of the effect induced by periodic forcing. As we move closer to the singularity ($d = 0.25$), the approximation fails, as it does at resonance $d = 0$. Therefore, we obtain the same qualitative behavior as in Example 1.

REMARK 10 (expansions near resonance). *A more detailed view of the situation near the resonance can be obtained by reexpanding the equations. More exactly, for*



(a) $M(x,v)$. (b) $N(x,v)$.

FIG. 5.6. *Examples of the numerical surfaces M, N for the nonlinear oscillator. M, N and other functions need to be differentiated in space to obtain numerical second order coefficients.*

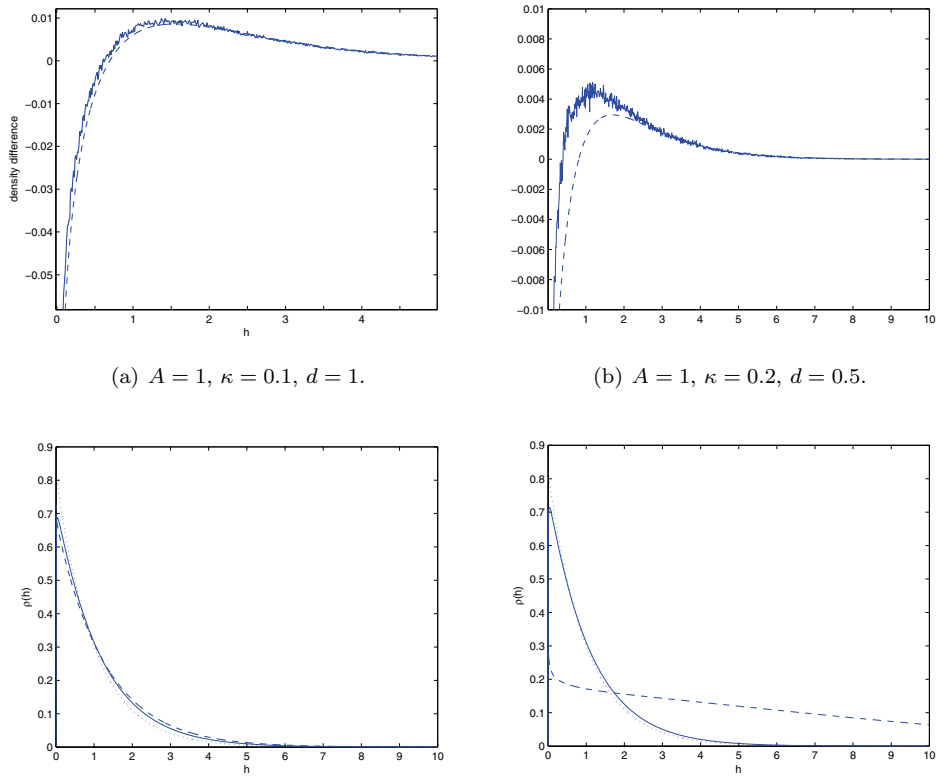


(a) Drift $a(h)$. (b) Diffusion $\sigma^2(h)$.

FIG. 5.7. *Drift and diffusion coefficients for the nonlinear example with $n = 2$, $A = 1$, $\epsilon = 1$, and $d = 1$. (solid -) $\kappa = 0.6$, (dashed - -) zeroth order case $\kappa = 0$.*

example in the linear case, one may expand $\omega = 1 + \epsilon\kappa\Omega$ and look for the resulting equations. Finally, the results obtained from such a procedure have to be matched to the results obtained in the present work away from the resonance. This will be left for future work.

REMARK 11 (resonance for constant forcing). *If we choose $c(x,v) := \omega = \text{const.}$, there exists a resonance point $\bar{h}$ where condition* (4.30) *is violated, and the asymptotic method in principle fails. However, the numerically obtained results strongly depend on the value of $\bar{h}$: If, for example, $\bar{h} \approx 1$, there is a singularity in the functions M, N which destroys all further results, and no meaningful coefficients can be obtained. If, on the contrary, $\bar{h}$ is small or very large, its induced singularity in M, N is hardly seen numerically, and the numerical higher order coefficient gives an approximation to the full system's stationary energy distribution of the same quality as in Examples* 1 *and* 2.



(a) $A = 1$, $\kappa = 0.1$, $d = 1$.

(b) $A = 1$, $\kappa = 0.2$, $d = 0.5$.

(c) $A = 1$, $\kappa = 0.6$, $d = 0.5$.

(d) $A = 1$, $\kappa = 0.1$, $d = 0.25$.

FIG. 5.8. *Stationary distribution of nonlinear oscillator $n = 2$ with different values of κ and d: $(\cdot\cdot)$ unforced stationary solution, $(-)$ full SDE system by MC-simulation, $(--)$ asymptotic results of higher order averaging. In* (a) *and* (b) *we plot the difference to the unforced stationary distribution, and in* (c) *and* (d) *we plot the distributions multiplied by T_h on a regular scale. For all cases $A = 1$, $\epsilon = 1$.*

6. Conclusion and outlook. In this paper, we have investigated a kinetic Fokker–Planck equation with periodic forcing. We have introduced a splitting ansatz projecting the probability density f on an energy-dependent density Pf and derived a system of equations for both Pf and its remainder Qf. With an approximate solution of Qf, higher order approximations of Pf have been computed. The method is constructed for situations in which the frequency of the forcing term does not correspond to the Hamiltonian motion of the unforced system. It is highly flexible in terms of deriving approximate equations for Hamiltonian energy processes, as it allows the investigation of different orders of disturbance as well as different orders of approximation. Higher order approximations are necessary, since despite the fact that the forcing affects the system and the energy dynamics, its contribution to the limiting energy equation obtained by stochastic averaging is usually visible only if coefficients are computed up to second order. We were able to derive first- and second order approximate equations away from resonances based on averaged coefficients, and a numerical scheme for their computation. In the two examples that have been investigated, the second order approximation shows a very accurate match with full

system dynamics for parameters away from resonance. Not only do we gain both an approximation of energy equilibrium and the corresponding relaxation, but also the method allows us to better understand the influence of the forcing on the energy dynamics, thanks to the higher order drift and diffusion coefficients. In future work, these results will be extended with additional expansions near resonance points to obtain a more detailed picture in the cases not covered by the current asymptotic expansion. Additionally, as we mentioned in the introduction, the present procedure will be applied to the treatment of an asymptotic problem for a model describing an industrial textile production process; see [6, 4].

Appendix A. Derivation of explicit coefficients for the harmonic oscillator. In this appendix we gather all computations for the higher order averaging of the harmonic oscillator in greater detail. We consider the system (2.5) and the associated Fokker–Planck equation (2.4), which we rewrite as $\partial_t f = \frac{1}{\epsilon}\mathcal{L}_2 f + \mathcal{L}_1 f$, where

$$\mathcal{L}_2 f = -v\partial_x f + x\partial_v f - \partial_\phi(\omega f), \tag{A.1}$$

$$\mathcal{L}_1 f = \kappa\partial_v(\cos\phi f) + \frac{A^2}{2}\partial_v(\partial_v f + vf). \tag{A.2}$$

These operators together with the splitting ansatz $f = Pf + \epsilon Qf$ give system (4.7). Using $\mathcal{L}_2 Pf = 0$, $\langle \mathcal{L}_2 f\rangle = 0$ and the quasi-stationarity assumption, we obtain

$$\partial_t p - \langle \mathcal{L}_1 Pf\rangle = \epsilon \langle \mathcal{L}_1 Qf\rangle, \tag{A.3}$$

$$-\mathcal{L}_2 Qf - \epsilon\mathcal{L}_1 Qf = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega} - \epsilon\frac{1}{2\pi\Omega}\langle \mathcal{L}_1 Qf\rangle. \tag{A.4}$$

With the assumptions of section 4.3, we obtain

$$\langle \mathcal{L}_1 Pf\rangle = -\frac{A^2}{2}\partial_h(\overline{(1-v^2)p}) + \frac{A^2}{2}\partial_h^2\left(\overline{v^2}p\right), \tag{A.5}$$

$$\langle \mathcal{L}_1 Qf\rangle = \frac{A^2}{2}\partial_h^2\langle v^2 Qf\rangle - \frac{A^2}{2}\partial_h\langle (1-v^2)Qf\rangle + \kappa\,\partial_h\langle v\cos\phi Qf\rangle. \tag{A.6}$$

Before we continue, let us remark that $\overline{x^2} = h$, $\overline{v^2} = h$, $\overline{x^2v^2} = \frac{1}{2}h^2$, $\overline{v^4} = \frac{3}{2}h^2$.

A.1. Zeroth order. The left-hand side of (A.3) is $\mathcal{O}(1)$, and the right-hand side is $\mathcal{O}(\epsilon)$ and neglected to this order. We obtain

$$-\langle \mathcal{L}_1 Pf\rangle = \frac{A^2}{2}\partial_h(\overline{(1-v^2)p}) - \frac{A^2}{2}\partial_h^2\left(\overline{v^2}p\right), \tag{A.7}$$

which, after computing the averages, gives the equation

$$\partial_t p + \frac{A^2}{2}\partial_h((1-h)p) = \frac{A^2}{2}\partial_h^2(hp). \tag{A.8}$$

These are the classic stochastic averaging coefficients.

A.2. First order. Including terms of $\mathcal{O}(\epsilon)$ means approximating (A.4) by $-\mathcal{L}_1 Pf = \mathcal{L}_2 Qf$. For the harmonic oscillator, this reads

$$-\kappa\partial_v(\cos\phi Pf) - \frac{A^2}{2}\partial_v(\partial_v Pf + vPf) = -v\partial_x Qf + x\partial_v Qf - \partial_\phi(\omega Qf). \tag{A.9}$$

Using $\partial_h Pf = -Pf$ eliminates the A-dependent part of the equation and yields

$$-\kappa\cos\phi\partial_v(Pf) = -v\partial_x Qf + x\partial_v Qf - \partial_\phi(\omega Qf). \tag{A.10}$$

Using the ansatz $Qf = \kappa Pf(M\cos\phi + N\sin\phi)$ leads to

$$\dot{x} = v, \tag{A.11a}$$

$$\dot{v} = -x, \tag{A.11b}$$

$$\dot{X} = i\omega X - v, \tag{A.11c}$$

with $X = M + iN$. The homogeneous solution to the third equation is $\mathrm{Hom}(t) = e^{i\omega t}$. One inhomogeneous solution can be computed as

$$\mathrm{Inhom}(t) = e^{i\omega t}\int e^{-i\omega s}(-v(s))\mathrm{d}s = e^{i\omega t}e^{-i\omega t}\frac{x - i\omega v}{\omega^2 - 1} = \frac{x - i\omega v}{\omega^2 - 1}. \tag{A.12}$$

Since $\mathrm{Inhom}(t)$ is periodic with period in $T_h = 2\pi$, we can choose $C = 0$, $X(t) = \mathrm{Inhom}(t) \Rightarrow M = \frac{x}{\omega^2-1}$, $N = \frac{-\omega v}{\omega^2-1}$. The resulting approximative solution of Qf is

$$Qf = \kappa Pf\left(\frac{x}{\omega^2 - 1}\cos\phi - \frac{\omega v}{\omega^2 - 1}\sin\phi\right). \tag{A.13}$$

Inserting this into $\langle L_1 Qf\rangle$, we immediately see from (A.6) that

$$\frac{A^2}{2}\,\partial_h\left\langle(1 - v^2)Qf\right\rangle - \frac{A^2}{2}\partial_h^2\left\langle v^2 Qf\right\rangle = 0, \tag{A.14}$$

since the average integral over $\mathrm{d}\phi$ is zero. For the remaining term of $\langle L_1 Qf\rangle$ we compute

$$\partial_h\left\langle v\cos\phi Qf\right\rangle = \kappa\partial_h\left(\overline{\frac{vx}{\omega^2 - 1}\cos^2\phi - \frac{\omega v^2}{\omega^2 - 1}\sin\phi\cos\phi p}\right) = 0, \tag{A.15}$$

since $\overline{\sin\cos} = 0$. Therefore, the first order approximation leads to no additional terms in the averaged coefficients. We have to proceed to second order.

A.3. Second order. Including terms of $\mathcal{O}(\epsilon^2)$ means solving (A.4) up to second order. Sorting by orders leads to the system

$$-\mathcal{L}_2 U_1 = \frac{1}{2\pi}\mathcal{L}_1\frac{p}{\Omega}, \qquad (\mathcal{O}(1)) \tag{A.16a}$$

$$-\mathcal{L}_2 U_2 = \mathcal{L}_1 U_1 - \left\langle\mathcal{L}_1 U_1\right\rangle. \qquad (\mathcal{O}(\epsilon)). \tag{A.16b}$$

The solution of (A.16a) is again $U_1 = \kappa Pf(\frac{x}{\omega^2-1}\cos\phi - \frac{\omega v}{\omega^2-1}\sin\phi)$, which gives, using $\partial_h Pf = -Pf$,

$$-\mathcal{L}_1 U_1 = \kappa^2\left(\cos^2\phi\frac{-xvPf}{w^2 - 1} + \sin\phi\cos\phi\frac{\omega Pf(v^2 - 1)}{w^2 - 1}\right) + \frac{\kappa A^2}{2}\left(\sin\phi\frac{\omega v Pf}{\omega^2 - 1}\right), \tag{A.17}$$

which means $G(M) = -xv\frac{1}{\omega^2-1}$, $H(N) = -(1-v^2)\frac{\omega}{\omega^2-1}$, $F(M) = 0$, $F(N) = v\frac{\omega}{\omega^2-1}$, and $\langle\mathcal{L}_1 U_1\rangle = 0$, which is dropped henceforth. Following the general notation, the above problem is split into two parts:

$$\mathcal{L}_2 U_2^\kappa = -(\mathcal{L}_1 U_1)^\kappa, \tag{A.18}$$

$$\mathcal{L}_2 U_2^{\kappa A} = -(\mathcal{L}_1 U_1)^{\kappa A}, \tag{A.19}$$

with $U_2 = U_2^{\kappa A} + U_2^\kappa$.

A.3.1. Computing $U_2^{\kappa A}$. The κA^2-dependent equation can be solved by the method of characteristics, and we obtain

$$\dot{x} = v, \tag{A.20a}$$

$$\dot{v} = -x, \tag{A.20b}$$

$$\dot{X^{\kappa A}} = i\omega X + iv\frac{\omega}{\omega^2 - 1}, \tag{A.20c}$$

with $X^{\kappa A} = M^{\kappa A} + iN^{\kappa A}$ and $U_2^{\kappa A} = \frac{A^2}{2}\kappa(M^{\kappa A}\cos\phi + N^{\kappa A}\sin\phi)Pf$. The homogeneous solution reads $\mathrm{Hom}(t) = e^{i\omega t}$. An inhomogeneous solution is computed as

$$\begin{aligned}\mathrm{Inhom}(t) &= e^{i\omega t}\int e^{-i\omega s} iv\frac{\omega}{\omega^2-1}\mathrm{d}s = e^{i\omega t}\frac{\omega}{\omega^2-1}\left[\frac{ie^{-i\omega t}(-x+i\omega v)}{\omega^2-1}\right]\\ &= \frac{\omega}{(\omega^2-1)^2}(-ix-\omega v).\end{aligned} \tag{A.21}$$

Therefore we can choose $C = 0$ (inhomogeneous solution is periodic) and obtain

$$M^{\kappa A} = -\frac{\omega^2}{(\omega^2-1)^2}v, \qquad N^{\kappa A} = -\frac{\omega}{(\omega^2-1)^2}x. \tag{A.22}$$

A.3.2. Computing U_2^{κ}. Following the general notation, we have to solve

$$\dot{X^{\kappa}} = i2\omega X - vx\frac{1}{\omega^2-1} + i(v^2-1)\frac{\omega}{\omega^2-1}, \tag{A.23a}$$

$$\dot{R^{\kappa}} = -\frac{1}{2(\omega^2-1)}xv, \tag{A.23b}$$

with $X^{\kappa} = M^{\kappa} + iN^{\kappa}$ and $U_2^{\kappa} = \kappa^2 Pf(M^{\kappa}(\cos^2\phi - \frac{1}{2}) + N^{\kappa}\cos\phi\sin\phi + R^{\kappa})$, $\overline{R^{\kappa}} \stackrel{!}{=} 0$. The homogeneous solution reads $\mathrm{Hom}(t) = e^{2i\omega t}$. An inhomogeneous solution is computed as

$$\begin{aligned}\mathrm{Inhom}(t) &= e^{2i\omega t}\int e^{-2i\omega s}\left(-vx\frac{1}{\omega^2-1} + i(v^2-1)\frac{\omega}{\omega^2-1}\right)\mathrm{d}s\\ &= \frac{1}{\omega^2-1}e^{2i\omega t}\left[e^{-2\omega t}\left(-\frac{2h(\omega^2+1)\cos(2t)}{4(\omega^2-1)} + \frac{(1-h)}{2} - \frac{h\omega i\sin(2t)}{\omega^2-1}\right)\right].\end{aligned} \tag{A.24}$$

Dropping the constant part and using $\cos(2t) = \cos^2(t) - \sin^2(t)$, $\sin(2t) = 2\sin(t)\cos(t)$, we obtain

$$\mathrm{Inhom}(t) = \frac{1}{(\omega^2-1)^2}\left(\frac{(\omega^2+1)\cdot(x^2-v^2)}{4} - i\omega xv\right). \tag{A.25}$$

Again, this is periodic; thus $C = 0$ and

$$M^{\kappa} = \frac{\omega^2+1}{4(\omega^2-1)^2}\left(x^2-v^2\right), \qquad N^{\kappa} = -\frac{\omega}{(\omega^2-1)^2}xv. \tag{A.26}$$

Furthermore we compute

$$R^{\kappa} = -\frac{1}{2}\int\frac{1}{(\omega^2-1)}vx\mathrm{d}t = \frac{1}{4(\omega^2-1)}v^2 + C. \tag{A.27}$$

To ensure $\overline{Qf} = 0$, we compute $C = -\frac{1}{4(\omega^2-1)}h$. Therefore

$$R^{\kappa} = \frac{1}{4(\omega^2-1)}(v^2-h). \tag{A.28}$$

A.3.3. Putting the results together. As a result of the above computations,

$$Qf = U_1 + \epsilon U_2 = U_1 + \epsilon(U_2^{\kappa A} + U_2^{\kappa}) \tag{A.29}$$
$$= \frac{\kappa}{w^2-1}\left(x\cos\phi - \omega v\sin\phi\right)Pf + \epsilon\frac{A^2\kappa}{2(w^2-1)^2}\left(-\omega^2 v\cos\phi - \omega x\sin\phi\right)Pf$$
$$+\epsilon\frac{\kappa^2}{(w^2-1)^2}\left(\frac{\omega^2+1}{4}\left(x^2-v^2\right)\left(\cos^2\phi-\frac{1}{2}\right) - \omega xv\cos\phi\sin\phi + \frac{\omega^2-1}{4}(v^2-h)\right)Pf.$$

This is now plugged into (A.3) using (A.6). In this way, we can eliminate many terms by a careful investigation. Averages with factor $\frac{A^2}{2}$: Qf is not multiplied by a function of ϕ, so all terms including $\cos\phi, \sin\phi, \cos\phi\sin\phi$ average to zero. In the first order approximation, all averages are zero. In the second order approximation, we have a term of $\cos^2\phi$ which does not average to zero. Averages with factor κ: We have an additional factor $\cos\phi$, so the only terms not averaging to zero are those having a term $\cos\phi$. Out of those, one averages to zero since $\overline{xv} = 0$. Alltogether, we obtain

$$\langle\mathcal{L}_1 Qf\rangle = \frac{\epsilon A^2\kappa^2}{2}\frac{1}{(\omega^2-1)^2}\left[\partial_{h^2}\left(\overline{\frac{\omega^2-1}{4}v^2(v^2-h)}p\right)\right.$$
$$\left. - \partial_h\left(\overline{\frac{\omega^2-1}{4}(1-v^2)(v^2-h)}p\right) + \partial_h\left(\overline{-\omega^2 v^2\cos^2\phi}p\right)\right] \tag{A.30}$$
$$= \frac{\epsilon A^2\kappa^2}{4}\frac{1}{4(\omega^2-1)}\partial_h(h^2 p) + \frac{\epsilon A^2\kappa^2}{4}\frac{1}{4(\omega^2-1)}\partial_{h^2}(h^2 p) - \frac{\epsilon A^2\kappa^2}{4}\frac{\omega^2}{(\omega^2-1)^2}\partial_h\left[hp\right].$$

The second order stochastic coefficients are therefore

$$a(h) = \frac{A^2}{2}(1-h) + \frac{\epsilon^2 A^2\kappa^2}{4}\left(\frac{\omega^2}{(\omega^2-1)^2}h - \frac{1}{4(\omega^2-1)}h^2\right), \tag{A.31}$$

$$\sigma^2(h) = A^2 h + \frac{\epsilon^2 A^2\kappa^2}{2}\frac{1}{4(\omega^2-1)}h^2. \tag{A.32}$$

A.3.4. Expansions of equilibrium. Here, we give details of the computations presented in section 4.3.2, Example 1. First, we look at the stationary Fokker–Planck equation of the linear oscillators

$$v\partial_x f - x\partial_v f - \epsilon\kappa\cos\phi\partial_v f = \frac{A^2}{2}\partial_v(\partial_v f + vf), \tag{A.33}$$

which can be rewritten as $v\partial_x f - (x - \epsilon\kappa\cos\phi)\partial_v f = \frac{A^2}{2}\partial_v(\partial_v f + vf)$. Thus, the new Hamiltonian is

$$H_\epsilon = \frac{v^2}{2} + \frac{x^2}{2} + \epsilon\kappa x\cos\phi, \tag{A.34}$$

and the corresponding equilibrium is

$$f_s(x,v) = e^{-H_\epsilon(x,v)} =: f(\epsilon,\cdot). \tag{A.35}$$

An expansion in ϵ around $\epsilon = 0$ gives

$$f_s(x,v) = e^{-H(x,v)}(1 + \epsilon\kappa x\cos\phi + \epsilon^2\kappa^2 x^2\cos^2\phi). \tag{A.36}$$

Using averages, i.e., replacing $x \to \overline{x} = 0$, $\cos\phi \to \overline{\cos\phi} = 0$, $x^2 \to \overline{x^2} = h$, $\cos^2\phi \to \overline{\cos^2\phi} = \frac{1}{2}$, gives

$$f_s(x,v) \approx e^{-H(x,v)}\left(1 + \frac{\epsilon^2\kappa^2}{4}H(x,v)\right), \qquad p_s(h) \approx e^{-h}\left(1 + \frac{\epsilon^2\kappa^2}{4}h\right). \tag{A.37}$$

The same result is obtained by setting $\omega = 0$ in the explicit higher order coefficients (5.5a), inserting into (5.6), and expanding in ϵ:

$$p_s(h) = \exp\left(\int_0^h \frac{-A^2h' + \frac{\epsilon^2A^2\kappa^2}{8}h'^2 + \frac{\epsilon^2A^2\kappa^2}{4}h}{A^2h' - \frac{\epsilon^2A^2\kappa^2}{8}h'^2}\mathrm{d}h'\right) \tag{A.38a}$$

$$= \exp\left(-h - 2\log\left(2 - \frac{\epsilon^2\kappa^2}{4}h\right) - 2\log 2\right) \tag{A.38b}$$

$$= \frac{4e^{-h}}{4 - \epsilon^2\kappa^2h + \frac{\epsilon^4\kappa^4}{16}h^2} =: g(\epsilon). \tag{A.38c}$$

Expanding this to second order in ϵ gives again $g(\epsilon) \approx e^{-h}\left(1 + \frac{\epsilon^2\kappa^2}{4}h\right)$.

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