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# THE ONSET AND END OF THE GUNN EFFECT IN EXTRINSIC SEMICONDUCTORS*

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**Abstract.** A Hopf bifurcation analysis of the spontaneous current oscillation in direct current (DC) voltage-biased extrinsic semiconductors is given for the classical model of the Gunn effect in n-GaAs. For semiconductor lengths $L$ larger than a certain minimal value, the steady state is linearly unstable for voltages in an interval $(\phi_\alpha, \phi_\omega)$. As $L$ increases, the branch of time-periodic solutions bifurcating at simple eigenvalues when $\phi = \phi_\alpha$ turns from subcritical to supercritical and then back to subcritical again. For very long semiconductors a quasi continuum of oscillatory modes bifurcates from the steady state at the onset of the instability. The bifurcating branch is then described by a scalar reaction-diffusion equation with cubic nonlinearity subject to antiperiodic boundary conditions on a subinterval of $[0, L]$. For the electron velocity curve we have considered, the bifurcation is subcritical, which may agree with experimental observations in n-GaAs. An extension of our calculation suggests that a supercritical time-periodic bifurcating branch (possible for other electron velocity curves) consists of the generation at $x = 0$ and evolution of waves that are damped before they can reach the receiving contact. Our calculation is a first step in determining how the bifurcating solution branch is related to the branch of oscillatory solutions mediated by solitary wave dynamics. The relation to previous numerical and experimental results is discussed.

**Key words.** Gunn effect, semiconductor instabilities, Hopf bifurcation, multiscale methods

**AMS subject classifications.** 35G25, 35M05, 35B25

## 1. Introduction.

Semiconductors in which spontaneous bulk current instabilities occur have been shown to exhibit a wide range of temporal oscillatory and chaotic behavior under suitable bias conditions, including period doubling and frequency locking routes to chaos in Ge [46], GaAs [1], and InSb [41]. These phenomena are observed by measuring the current in the external circuit connected to the semiconductor and the electric field (or the electric potential) inside the semiconductor. Often [22], they are caused by the dynamics of one-dimensional nonlinear waves of electric charge inside the semiconductor and their interaction with the ohmic contacts at its boundary. The simplest case seems to be that of the Gunn instability [19], a periodic oscillation of the current through a purely resistive external circuit under direct current (DC) voltage bias. The oscillations are caused by the periodic generation of charge domains (solitary waves) at one contact, their uniform motion inside the semiconductor, and their annihilation at the other contact [2], [22], [21], [42]–[44].

Although the physics of the Gunn instability in, say, n-GaAs has been well understood for years [42]–[44], the mathematical understanding of the phenomenon is not complete. In previous publications we have tried to fill this gap [2], [22], [6], [8]. A reasonable model for the dynamics of the electric field and the electric current in n-GaAs is a nonlinear parabolic equation (with small diffusivity) coupled to an integral conservation law (the DC voltage bias) [11], [31], [42]. These equations

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are

$$\epsilon_s \frac{\partial \tilde{E}}{\partial \tilde{t}} + \mu \tilde{E}_R\, v\left(\frac{\tilde{E}}{\tilde{E}_R}\right)\left[q\tilde{n}_D + \epsilon_s \frac{\partial \tilde{E}}{\partial \tilde{x}}\right] - qD\,\frac{\partial^2 \tilde{E}}{\partial \tilde{x}^2} = \tilde{J}_{tot}, \tag{1.1}$$

$$\int_0^{\tilde{L}} \tilde{E}(\tilde{x},\tilde{t})\,d\tilde{x} = \tilde{\phi}. \tag{1.2}$$

Here the unknowns are $\tilde{E}(\tilde{x},\tilde{t})$, the electric field inside the semiconductor, and $\tilde{J}_{tot}(\tilde{t})$, the total electric current (essentially the same current measured on the external circuit). That $\tilde{J}_{tot}(\tilde{t})$ does not depend on the distance $\tilde{x}$ is a feature of the one-dimensional geometry: (1.1) is obtained by integrating once with respect to $\tilde{x}$ the continuity equation for the charge density [2], [22] in a unipolar drift-diffusion model [36]. (1.1) is Ampère's law that establishes that the sum of the displacement current, $\epsilon_s\partial\tilde{E}/\partial\tilde{t}$, and the electron flux at a point of the semiconductor, $\mu\tilde{E}_R v(\tilde{E})(q\tilde{n}_D + \epsilon_s\partial\tilde{E}/\partial\tilde{x}) - qD\partial^2\tilde{E}/\partial\tilde{x}^2$, is equal to the total current, $\tilde{J}_{tot}(\tilde{t})$. $\epsilon_s$, $q$, $\mu$, and $\tilde{E}_R$ are the permittivity, electron charge, mobility, and a reference electric field, respectively. The electron flux is the sum of drift and diffusion terms, which are, respectively, $q\mu\tilde{E}_R v(\tilde{E}/\tilde{E}_R)\tilde{n}$ and $-qD\partial\tilde{n}/\partial\tilde{x}$, where the electron density is $\tilde{n} = \tilde{n}_D + (\epsilon_s/q)\,\partial\tilde{E}/\partial\tilde{x}$ by Poisson's law. ($\tilde{n}_D$ is the constant concentration of donor impurities.) Equation (1.2) is the DC voltage bias condition that establishes the total voltage across the semiconductor to be $\tilde{\phi}$. (We are ignoring the contact built-in potential and the resistance on the external circuit, whose effects can be incorporated straightforwardly in our considerations.) While more complicated models also display Gunn oscillations [12] [37], the present model is generally agreed to be sufficient to account for the experimental observations in semiconductors longer than 1 $\mu$m [42], [44]. It is convenient to work with dimensionless versions of equations (1.1) and (1.2) [2], [22]:

$$\frac{\partial E}{\partial t} + v(E)\left(1 + \frac{\partial E}{\partial x}\right) - \delta\,\frac{\partial^2 E}{\partial x^2} = J, \qquad t > 0,\ 0 < x < L, \tag{1.3}$$

$$\int_0^{L} E(x,t)\,dx = \phi. \tag{1.4}$$

Here $E = \tilde{E}/\tilde{E}_R$, $x = q\tilde{n}_D\tilde{x}/(\epsilon_s\tilde{E}_R)$, $t = q\mu\tilde{n}_D\tilde{t}/\epsilon_s$, $J = \tilde{J}_{tot}/(q\mu\tilde{n}_D\tilde{E}_R)$, $\phi = q\tilde{n}_D\tilde{\phi}/\epsilon_s$, and $\delta = qD\tilde{n}_D/(\epsilon_s\mu\tilde{E}_R^2)$. Equations (1.3) and (1.4) are to be solved with convenient boundary and initial conditions:

$$E + \rho\left(\frac{\partial E}{\partial t} - J\right) = 0 \quad \text{at } x = 0, L;\ t > 0, \tag{1.5}$$

$$E(x,0) = f(x) \geq 0, \quad 0 < x < L. \tag{1.6}$$

Equation (1.5) is Ohm's law for the metal–semiconductor contacts: the field at the contact is proportional to the electron current (equal to $J(t) - \partial E/\partial t$ according to (1.3)). The proportionality constant is the dimensionless contact resistivity, $\rho \geq 0$, which we have assumed to be the same for both contacts at $x = 0$, $L$. (Unequal contact resistivities produce only obvious changes in our results.) The idea of using a boundary condition at the contacts that locally relates the electron current with the field is from Kroemer [32]. Contact current density–electric field relationships more general than the linear law (1.5) have been used by Grubin [17]. Equations (1.3)–(1.6) together with a $v(E)$ curve that grows at most linearly as $E \to \infty$ constitute a mathematically well-posed problem: there is a unique classical solution which depends smoothly on the initial data and parameters [34].

The peculiar physics of n-GaAs introduces the requirement that the velocity curve have negative slope (negative differential mobility, NDM) for fields on a certain interval $(E_M, E_m)$; see Fig. 1. This fact is one of the crucial conditions for the instability of steady states which causes the Gunn effect [40]. (The other crucial conditions are the voltage bias (1.4) and the boundary conditions (1.5) [42].) A convenient phenomenological expression for $v(E)$ is

$$v(E) = \frac{1 + B\,E^4}{1 + E^4}\,E, \qquad 0 < B \ll 1, \tag{1.7}$$

which was introduced by Kroemer in dimensional form [31]. $B$ is the ratio between the electron mobilities at the principal and satellite valleys of the conduction band of the semiconductor.

The dimensionless diffusivity $\delta > 0$ turns out to be very small [2], and the dimensionless sample length $L$ ranges from 2 to 450 (which corresponds to lengths from 10 to 200 $\mu$m for GaAs with an impurity concentration of $10^{15}$ cm$^{-3}$ [2], [19], [44]. Thus the limit $\delta \to 0$, $L \to \infty$ describes the experimental situation in long semiconductors and has been exploited to construct asymptotic solutions of the problem (1.3)–(1.7) [2], [6], [22], [8]. Most of the essential features of the asymptotic (stationary and time-dependent) solutions are already present in the outer problem $\delta = 0$ (with appropriate shock conditions [29]). This applies to the asymptotic description of the time-periodic Gunn oscillations (including creation of waves at the injecting contact, motion toward $x = L$, annihilation there, and recycling [22]) and to the onset and end of the Gunn instability as we will explain here (see also [8]). Szmolyan [45] (see also [36, § 4.8]) has also studied the system of equations (1.3)–(1.4) with a different scaling and with different boundary conditions. In particular the stage of one period

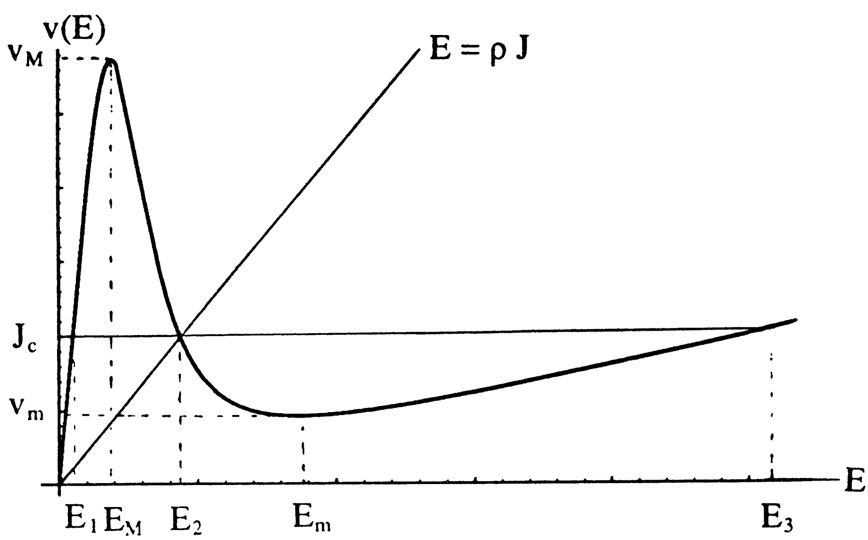


FIG. 1. *Electron velocity versus electric field* (1.7) *with* $B = 0.02$. $v_M = v(E_M)$, $v_m = v(E_m)$.

of the Gunn oscillations when a solitary wave is far from the contacts was analyzed on the basis of the Butcher integral representation of the Gunn domain [11], [29]. An asymptotic description of the processes of wave annihilation at the receiving contact and of wave creation at the injecting contact was not provided.

In the limit $\delta \to 0$ we can resort to matched asymptotic expansions to understand the Gunn effect. The positive sign of the convective term in (1.3) implies that disturbances propagate from left to right. This in turn indicates that a boundary layer appears at the receiving contact $x = L$ [6], [22], [8]. Setting $\delta = 0$ in (1.3) and ignoring the boundary condition at $x = L$, we obtain the reduced outer equation

$$\frac{\partial E}{\partial t} + v(E)\left[1 + \frac{\partial E}{\partial x}\right] = J, \qquad t > 0,\ \ 0 < x < L, \tag{1.8}$$

to be solved together with (1.4) and appropriate initial data (1.6) and the boundary condition

$$E(0,t) + \rho\left(\frac{\partial E(0,t)}{\partial t} - J(t)\right) = 0, \qquad t > 0, \tag{1.9}$$

Notice that the outer problem is only seemingly hyperbolic: a localized disturbance of the field at a certain point immediately influences $E(x,t)$ at any other points $x \in (0, L)$ via the integral constraint (1.4).

Near $x = L$, we must insert a diffusive quasi-stationary boundary layer [2], [6] (see the discussion in [8]), with $E = E_{in}(x,t)$:

$$v(E_{in})\frac{\partial E_{in}}{\partial x} \sim \delta\frac{\partial^2 E_{in}}{\partial x^2}, \tag{1.10}$$

$$E_{in}(L,t) = E_c(t) \quad \text{with } \frac{dE_c}{dt} + \frac{E_c}{\rho} = J, \tag{1.11}$$

$$E_{in}(x,t) \sim E(x,t) \quad \text{as } (L - x) \gg \delta. \tag{1.12}$$

(1.11) is the boundary condition at $x = L$, and (1.12) is the matching condition with the solution of the outer problem, which we denote by $E(x,t)$ as in (1.8). The solution of (1.10)–(1.12) is

$$\frac{\partial E_{in}}{\partial \xi} = Q(E_{in}, E), \qquad \xi \equiv \frac{x - L}{\delta}, \tag{1.13}$$

$$Q(E_{in}, E) = \int_E^{E_{in}} v(s)ds, \tag{1.14}$$

$$\xi = \int_{E_c(t)}^{E_{in}(\xi,t)} \frac{ds}{Q(s,E)}. \tag{1.15}$$

Some insight on the nature of the problem (1.3)–(1.6) is obtained by relating it to a mathematically simpler problem [2], [6]. In fact, we may consider two mathematical problems associated to eq. (1.3), each corresponding to a realizable physical situation:

1. The direct problem, corresponding physically to current bias conditions, consists of solving (1.3), (1.5), and (1.6) for a known, fixed $J(t)$. Once $E$ is known, the voltage can be determined by using (1.4).

2. The inverse problem, corresponding physically to voltage bias conditions, consists of solving (1.3), (1.5), and (1.6) for an unknown $J(t)$ selected so that (1.4) holds. We call this the inverse problem since the "equation" (1.3) contains an unknown term, $J(t)$, whose determination is part of the problem.

The direct problem was considered in [2], [6], where its steady states were constructed and their linear stability was discussed with the help of a general theorem proved in [7]. The knowledge of the direct problem may help to solve the inverse problem. For example, we discussed the shape of the current-voltage characteristic diagram for the stationary solutions of the inverse problem on the basis of our knowledge of the direct problem, [2], [6], [22]. The linear stability properties of the stationary solutions of direct and inverse problems are related through the principle of the argument applied to the differential impedance (see the discussion in [6]). However, this relation is usually not a simple one-to-one correspondence between the critical voltage (above which the stationary solution of the inverse problem is linearly unstable) and the critical current (above which the stationary solution of the direct problem is linearly unstable) [6]. As a consequence, it is often more practical to consider directly the inverse problem as a problem for two unknowns $E(x,t)$ and $J(t)$ which have equal status, and this was the point of view adopted in [8]. There we studied the stability of the steady state of the voltage bias problem directly; we showed that it is linearly stable for $\phi$ outside a voltage interval $(\phi_\alpha, \phi_\omega)$ (for $L$ large enough). When $\phi$ is slightly larger than $\phi_\alpha$ or smaller than $\phi_\omega$, a quasi continuum of oscillatory modes becomes unstable in the limit $\delta \to 0,\ L \to \infty$, and explicit expressions for the corresponding frequencies of the modes, for $\phi_\alpha,\ \phi_\omega$, and for the dispersion relation can be found [8].

In this paper we analyze the Hopf bifurcations at $\phi_\alpha$ and $\phi_\omega$ in the limit $\delta \to 0$ for contact resistivities $\rho \in (E_M/v_M, E_m/v_m)$. The latter restriction implies that the field near the injecting contact $x = 0$ at the steady state decreases with $x$ for low voltages. This gives rise to a potential drop, which is reasonable to expect in view of the built-in potential induced by a metal–semiconductor contact [44]. Besides, it is known that the Gunn effect is then mediated by solitary wave dynamics [22]. For smaller $\rho$, the Gunn effect is mediated by monopole wavefronts [22] (not observed experimentally), while for $\rho > E_m/v_m$ the present model is probably unrealistic and has not been studied. We find amplitude equations both for $L = O(1)$ and for $L \gg 1$. A physical interpretation of the results in the later limit allows us to understand the numerical and experimental results known for years [19], [42], [11], showing periodic recycling of solitary waves that disappear before reaching the receiving contact. We also give plausible scenarios where the experimentally observed intermittency [23], [24] could be understood within the context of our model or of related ones [10], [3], [4].

The rest of this paper is organized as follows. In § 2 we revise the construction of the stationary states of (1.3)–(1.5) in the outer limit $\delta \to 0$ and their stability under DC voltage bias [8]. More explicit results are found in the limit $L \to \infty$, $\delta = o(1/L)$, including the dispersion relation for the quasi continuum of unstable oscillatory eigenvalues [8]. In § 3 we derive the key nonresonance condition that ensures the absence of secular terms in our perturbation expansions of later sections. Section 4 contains a multiscale calculation of the Hopf bifurcation for the Gunn effect at finite length, $L = O(1)$, which is one of the main results in this paper. The corresponding study for $L \gg 1$ is made in § 5. We derive an amplitude equation for the oscillatory part of the current and interpret its relevant solutions in terms of waves of the electric field that die before reaching $x = L$. Section 6 contains a discussion of our results and an attempt to understand the possible bifurcation diagrams of the Gunn instability. We also indicate how our results may help interpreting relevant experiments and point out several open problems. The appendices are devoted to several technical matters.

**2. Steady states and their linear stability.** This topic has already been considered in a number of publications [8], [6], [2]. We recall several results of [8] for resistivities $\rho \in (E_M/v_M, E_m/v_m)$.

When $E$ and $J$ are time independent, eqs. (1.8) and (1.9) become

$$\frac{\partial E}{\partial x} = \frac{J - v(E)}{v(E)}, \tag{2.1}$$

$$E(0) = \rho\, J. \tag{2.2}$$

Let $E(x; J)$ denote the solution of (2.1)–(2.2) corresponding to a given positive $J$. The current $J$ is then determined as a function of the applied voltage $\phi$ by the equation

$$\Phi(J) \equiv \int_0^L E(x; J)\, dx = \phi, \tag{2.3}$$

which has a unique solution $J$ for every positive $\phi$, [8]. The qualitative behavior of $E(x; J)$ and of $\Phi(J)$ is seen by the analysis of the one-dimensional phase diagram of (2.1). Its fixed points satisfy $J - v(E) = 0$. When the curve $v(E)$ is as in Fig. 1 and $v_m < J < v_M$, the function $J - v(E)$ has three zeroes $E_1(J) < E_2(J) < E_3(J)$. Considering (2.1) as a one-dimensional dynamical system, the critical point $E_2$ is unstable, while $E_1$ and $E_3$ are attractors with basins $E < E_2$ and $E > E_2$, respectively. The asymptotic value of $E(x; J)$ as $x$ increases (on long enough samples) is $E_1(J)$ or $E_3(J)$, depending on whether the initial point $(E, J)$ at $x = 0$ (which lies on the line $E(0) = \rho J$) is to the left or to the right of $E_2(J)$, respectively. Of particular interest is the value $J = J_c$ for which these two coincide:

$$E_2(J_c) = \rho\, J_c. \tag{2.4}$$

This relation holds only for resistivities $\rho \in (E_M/v_M, E_m/v_m)$, which we assume here as explained in § 1.

Let us determine the dependence of the current $J$ with the average field $\phi/L$ for the problem (2.1)–(2.3) in the limit $L \to \infty$. For $0 < J < J_c$, the field of the steady state decreases from $E(0; J) = \rho\, J$ down toward its asymptotic value $E_1(J)$, whereas for $J > J_c$, $E(x; J)$ monotonically increases from $E(0; J) = \rho\, J$ to $E_3(J)$ for large $x$ (see Fig. 2). In the limit $L \to \infty$, the voltage $\Phi(J)$ of (2.3) is then approximately $E_1(J)\, L$ for $0 < J < J_c$, and $E_3(J)\, L$ for $J > J_c$. We then have that

$$J \sim v\left(\frac{\phi}{L}\right) \tag{2.5}$$

both for $0 < \phi/L < E_1(J_c)$ and for $E_3(J_c) < \phi/L$ approximately as $L \to \infty$. The steady-state current therefore follows the curve $v(E)$ outside a voltage interval such that $E_1(J_c) < \phi/L < E_3(J_c)$. Inside this interval, $J \sim J_c$. The steady-state field corresponding to such voltages is described in the following lemma, which was proved in [8].

LEMMA 1. (a) *Let $E_1(J_c) < \phi/L < E_2(J_c)$. Let us fix $E(X; J) = E_0$, where $E_0$ is any convenient fixed number in the interval $E_1(J_c) < E_0 < E_2(J_c)$. Then for each*

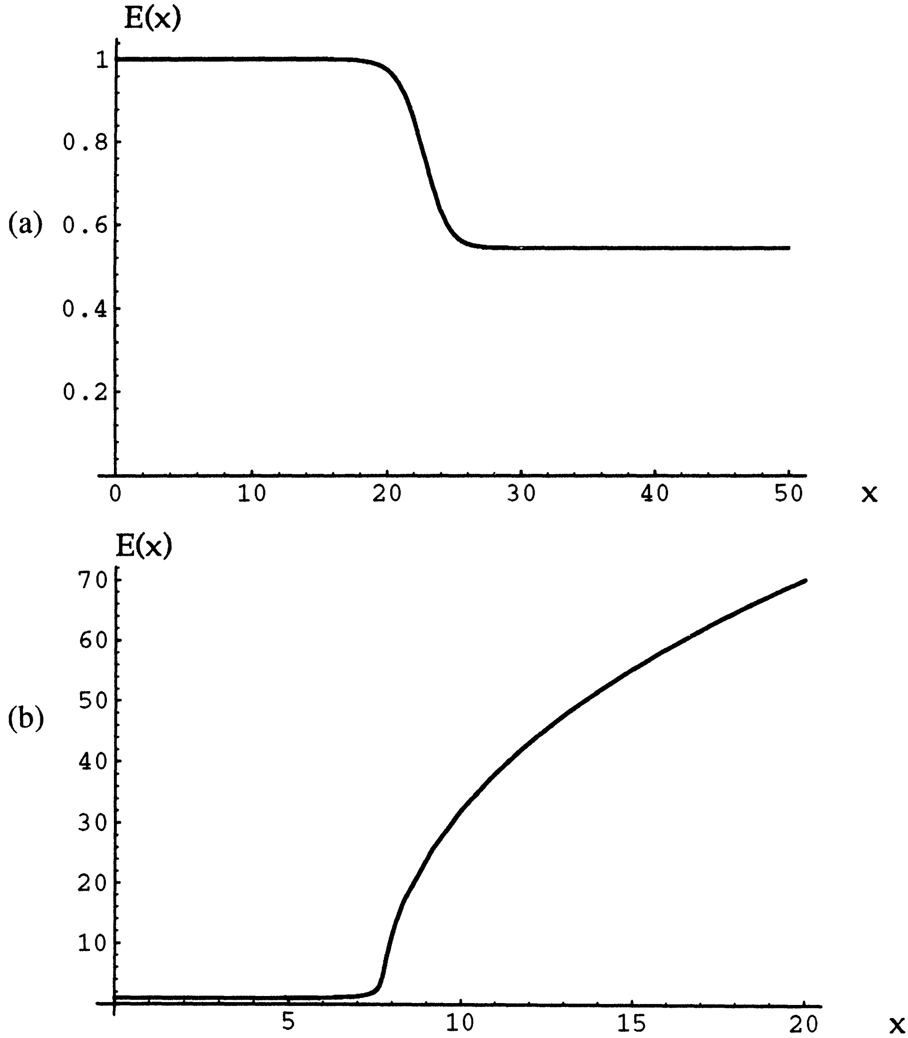


FIG. 2. *Steady state $E(x;J)$ when:* (a) $E_1(J_c)\,L < \phi < E_2(J_c))\,L$, $J \nearrow J_c$ *(numerical values:* $J_c - J = 5.01 \times 10^{-4}$; $\rho = 2$) *and* (b) $E_2(J_c)\,L < \phi < E_3(J_c)\,L$, $J \searrow J_c$ *(numerical values:* $J - J_c = 10^{-4}$; $\rho = 2$). *We have chosen a large enough $L$ and the narrow boundary layer at $x = L$ has been omitted.*

*value of $X \in (0, L)$ such that $\{X, (L - X)\} \gg 1$, $\phi$ and $J$ are uniquely determined by the asymptotic formulae*

$$\begin{aligned}\phi = {} & E_1 L + (E_2 - E_1)X + \left(\frac{L - X}{v_1'} + \frac{X}{v_2'}\right)(J - J_c) \\ & + \int_{E_2}^{E_0} \frac{(s - E_2)\,v(s)}{J_c - v(s)}\,ds + \int_{E_0}^{E_1} \frac{(s - E_1)\,v(s)}{J_c - v(s)}\,ds + o(1)\end{aligned} \tag{2.6}$$

*and*

$$J - J_c \sim -\frac{c_L \mid v_2' \mid}{1 - \rho v_2'} \exp\left\{-\frac{\mid v_2' \mid X}{J_c}\right\}. \tag{2.7}$$

*Also*

$$E(x;J) \sim E_2 - c_L \exp\left\{-\frac{v_2'(x-X)}{J}\right\} \quad (as\ \ (x-X) \to -\infty) \tag{2.8}$$

*with*

$$c_L = (E_2 - E_0)\exp\left\{\int_{E_0}^{E_2}\left[\frac{1}{s-E_2} - \frac{v_2'}{v(s)-J} - \frac{v_2'}{J}\right] ds\right\} \tag{2.9}$$

*and*

$$E(x;J) \sim E_1 - c_R \exp\left\{-\frac{v_1'(x-X)}{J}\right\} \quad (as\ \ (x-X) \to +\infty) \tag{2.10}$$

*with*

$$c_R = (E_0 - E_1)\exp\left\{-\int_{E_1}^{E_0}\left[\frac{1}{s-E_1} - \frac{v_1'}{v(s)-J} - \frac{v_1'}{J}\right] ds\right\}. \tag{2.11}$$

*In the right-hand sides of* (2.8) *and* (2.10), $v_k' \equiv v'(E_k(J))$, $k = 1, 2$. $E(x;J)$ *in these equations refers to the field in the transition layer* $(x - X) = O(1)$: $(x - X) \to -\infty$ (*resp.*, $\to +\infty$) *means leaving the transition layer toward the left* (*resp., right*). *Note that* $c_L$ *and* $c_R$ *are both positive.*

(b) *Let* $E_2(J_c) < \phi/L < E_3(J_c)$. *Let us fix* $E(X;J) = E_0$, *where* $E_0$ *is a given number in the interval* $E_2(J_c) < E_0 < E_3(J_c)$. *Then for each value of* $X \in (0, L)$ *such that* $\{X, (L - X)\} \gg 1$, $\phi$ *and* $J$ *are uniquely determined by the asymptotic formulae* (2.6)–(2.11), *where* $E_3(J)$ *replaces* $E_1(J)$ *everywhere. Note that* $c_L$ *and* $c_R$ *are now both negative.*

To analyze the linear stability of the steady state, we study the evolution of a small disturbance about the steady state

$$\begin{aligned} J(t) &= J + \epsilon\,\tilde{j}(t), \\ E(x,t) &= E(x) + \epsilon\,\tilde{e}(x,t), \qquad 0 < \epsilon \ll 1, \end{aligned} \tag{2.12}$$

as $t \to +\infty$. Inserting (2.12) into (1.8), (1.4), and (1.9) yields

$$\mathcal{L}\tilde{e} - \tilde{j} = 0. \tag{2.13}$$

$$\tilde{e}(0,t) + \rho\left(\frac{\partial \tilde{e}(0,t)}{\partial t} - \tilde{j}(t)\right) = 0, \quad t > 0, \tag{2.14}$$

$$\int_0^L \tilde{e}(x,t)\,dx = 0. \tag{2.15}$$

In (2.13), $\mathcal{L}$ is the operator

$$\mathcal{L}\tilde{e} \equiv \frac{\partial \tilde{e}}{\partial t} + \frac{\partial[v(E)\tilde{e}]}{\partial x} + v'(E)\tilde{e}. \tag{2.16}$$

That considering a small nonzero diffusivity $0 < \delta \ll 1$ does not modify our stability results was shown in Appendix B of [8]. Equations (2.13)–(2.15) can be solved by separation of variables:

$$\tilde{j}(t) = \hat{j}\,e^{\lambda t}, \ \ \tilde{e}(x,t) = \hat{e}(x;\lambda)\,e^{\lambda t}. \tag{2.17}$$

Insertion of (2.17) into (2.13)–(2.15) yields

$$\frac{\partial[v(E)\,\hat{e}]}{\partial x} + [v'(E) + \lambda]\,\hat{e} = \hat{j}, \tag{2.18}$$

$$Z(\lambda) \equiv \int_0^L \frac{\hat{e}(x;\lambda)}{\hat{j}}\,dx = 0, \tag{2.19}$$

$$\hat{e}(0;\lambda) = \frac{\rho\hat{j}}{1+\rho\lambda}. \tag{2.20}$$

The zero of the impedance $Z(\lambda)$ with largest real part determines the linear stability of the steady state. We have evaluated numerically the neutral stability curve (corresponding to the zero with largest real part being pure imaginary) for the steady state in the parameter space $\phi/L$ *vs* $L$ for different values of the resistivity $\rho$. The results are shown in Fig. 3. The discontinuities in the slope of the curve in Fig. 3 are due to the crossing of different zeroes as $L$ increases. Note that above a certain $L = L_m$, there are two values of the voltage for each $L$, $\phi_\alpha$, and $\phi_\omega$, such that the steady state is linearly stable for $\phi$ outside $(\phi_\alpha, \phi_\omega)$. At the voltages $\phi_\alpha$ and $\phi_\omega$, we

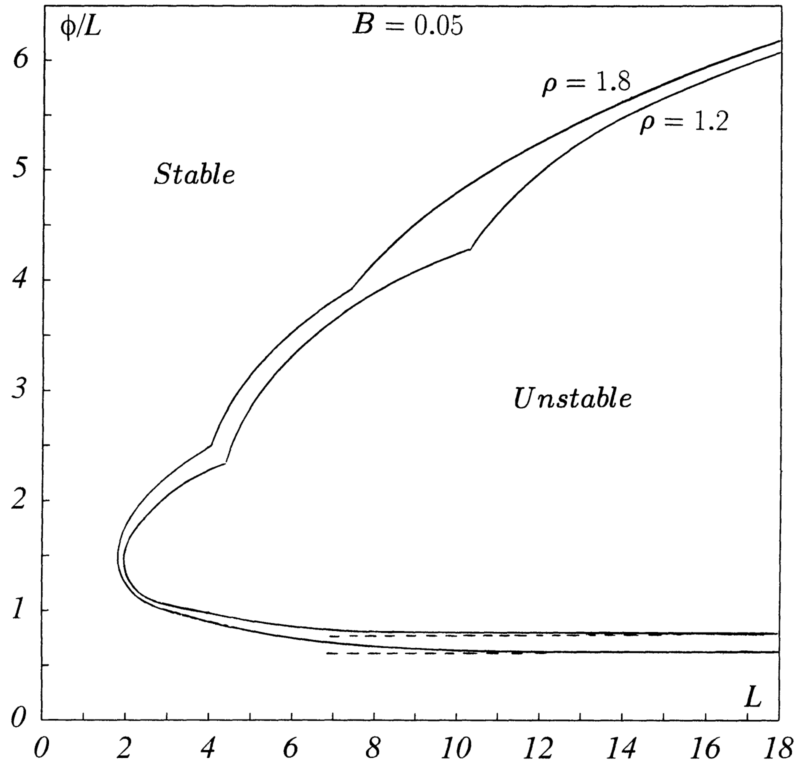


FIG. 3. *Neutral stability curve of the steady state for* $B = 0.05$ *and two values of* $\rho$. *The steady state may be linearly unstable to the right of the minimal length* $L_m$. *Note that* $L_m$ *increases with the resistivity of the injecting contact* $\rho$. *For each* $L > L_m$, *the steady state is linearly stable outside a voltage interval* $(\phi_\alpha, \phi_\omega)$: $\phi_\alpha$ *corresponds to the lower branch of the neutral stabilıty curve and* $\phi_\omega$ *to the upper branch. The dotted lines indicate the value* $E_1(J_c)$ *to which the lower branch of the neutral stability curve tends as* $L \longrightarrow \infty$. *The upper branch of the neutral stability curve tends to* $E_3(J_c)$ *as* $L \longrightarrow \infty$, *but a good approximation to this asymptotic value occurs only for much larger values of* $L$ *than those represented in the figure.*

expect that time-periodic solutions bifurcate from the steady state. We shall calculate these bifurcating branches of periodic solutions in §§ 3 and 4 for semiconductors of finite length. In the limit $L \gg 1$ many modes become unstable almost simultaneously, as we shall see below, and a different calculation of the bifurcating branches is needed; see § 5.

*Remark* 1. Note that $\phi_\alpha/L$ rapidly tends to a constant value $E_1(J_c)$ as $L$ increases. This suggests that the results we will obtain next in the asymptotic limit $L \to \infty$ may be of practical applicability even for moderate $L$ near $\phi = \phi_\alpha$.

For long semiconductors the linear stability of the steady state can be ascertained without resorting to numerical calculations, as was done in [8]. The main results are contained in Lemma 2 below, which was also proven in [8].

LEMMA 2. *Asymptotically as $L \to \infty$, the steady state is linearly unstable for voltages $\phi \in (\phi_\alpha, \phi_\omega)$, where*

(a) *$\phi_\alpha$ is given by* (2.6) *with $X = X_c$,*

$$X_c \sim \frac{J}{|\, v_2' \,|} \ln \left\{ \frac{c_L \, v_2'^2 \, L}{J v_1' (1 - \rho\, v_2')(E_2 - E_1)} \right\} = \frac{J}{|\, v_2' \,|} \ln L + O(1). \tag{2.21}$$

*For voltages slightly over $\phi_\alpha$, $\phi = \phi_\alpha + \delta\phi$,*

$$\frac{1}{(\ln L)^2} \ll \delta\phi \ll 1 \quad (as \ \ln L \to \infty), \tag{2.22}$$

*$X = X_c + \delta X$ (with $\delta X \sim \delta\phi/[2(E_2 - E_1)] > 0$), many eigenvalues $\lambda = i\Omega_n + \alpha_n + i\omega_n$, $(\alpha_n, \omega_n) \ll \Omega_n \ll 1$, become unstable, where*

$$\Omega_n \sim \frac{n\, J\, \pi}{X_c}, \quad n = \pm 1, \pm 3, \pm 5, \ldots, O(X_c \sqrt{\delta X}), \tag{2.23}$$

$$\alpha_n \sim \frac{|\, v_2' \,|\ \delta X}{X_c} - \frac{J\,(f_2 - \frac{f_1^2}{2})\, \Omega_n^2}{X_c}, \tag{2.24}$$

$$\omega_n \sim \frac{J\, f_1\, \Omega_n}{X_c}. \tag{2.25}$$

*Here the $f_k$'s are*

$$f_1 = -\rho - \frac{v_1' - v_2'}{v_1' \, v_2'} - \phi_1, \tag{2.26}$$

$$f_2 = (\rho + \phi_1) \left( \rho + \frac{v_1' - v_2'}{v_1' \, v_2'} \right) + \frac{v_1' - v_2'}{v_1' \, v_2'} - v_1' - \phi_2, \tag{2.27}$$

*and*

(2.28)

$$\phi_k = - \frac{(-1)^k}{k!\, (E_2 - E_1)} \int_{-\infty}^{+\infty} \frac{\partial E(x;J)}{\partial x} \left( \frac{x - X + E(x;J) - E_2}{J} \right)^k dx, \ \ k = 1, 2, \ldots .$$

(b) *$\phi_\omega$ is given by the formulae* (2.6) *and* (2.21) *with $E_3$ replacing $E_1$. With this replacement, formulae* (2.23)–(2.29) *hold for voltages $\phi = \phi_\omega - \delta\phi$, $X = X_c + \delta X$ ($\delta X \sim \delta\phi/[2(E_3 - E_2)] > 0$).*

*Remark* 2. The main elements of the proof of this lemma will be recalled in Appendix A, as we shall need them to motivate our derivation of the amplitude equation for long semiconductors in § 5. Lemma 2 confirms that a steady state with only a small fraction ($L^{-1}\ln L$) of its length in the NDM region of $v(E)$ may become unstable, as concluded by Grubin, Shaw, and Solomon [18] after a calculation with a piecewise linear $v(E)$ [42]. Steady states with a larger fraction of their length in the NDM region of $v(E)$, $X > X_c$, are unstable. Lemma 2 shows that these criteria for the onset of instability hold for general $v(E)$ in long semiconductors. From (2.21)–(2.23) and the formulas in Appendix A, we find that the eigenfunction corresponding to the eigenvalue with zero real part obeys

$$\frac{\hat{e}(x;i\Omega_n)}{\hat{j}} \sim \frac{L}{v_1'\,(E_2-E_1)}\,\frac{\partial E}{\partial x}\,\exp\left[-i\Omega_n\,\frac{x-X+E(x;J)-E_2}{J}\right] \tag{2.29}$$

on $0 < x < X_c + \Delta x$ and

$$\frac{\hat{e}(x;i\Omega_n)}{\hat{j}} \sim \frac{1}{v_1' + i\Omega_n} \qquad \text{on } X_c + \Delta x < x < L. \tag{2.30}$$

Here $1 \ll \Delta x \ll X_c = O(\ln L)$, and $X = X_c + \delta X$ such that $\mathrm{Re}\lambda = 0$, $\lambda \sim i\Omega_n$. Recalling the separation-of-variables ansatz, (2.17), we see that the disturbance of the electric field, $\tilde{e}(x,t) = \hat{e}(x;i\Omega_n)\exp(i\Omega_n t)$, represents a wave that travels with speed $J$ to the right while its amplitude grows exponentially with $x$ (recall (2.8)).

*Remark* 3. Equation (2.24) is a dispersion relation between the rate of growth of a given eigenvalue, $\delta X$, and the square of the imaginary part of the eigenvalue (the frequency). This equation relates the Fourier coefficients of the terms in a diffusion equation where the "spatial" variable corresponds to a slow time[1] scale $\chi = t\sqrt{\delta X}$ and the "time" variable corresponds to a slower time scale $\tau = (\delta X/X_c)\,t$. Then the coefficient of $-\Omega_n^2$ plays the role of an effective diffusivity, and it appears as such in the nonlinear amplitude equation that we will derive in § 5. For values of the resistivity in the interval $\rho \in (E_M/v_M, E_m/v_m)$, and typical values of $B$, the effective diffusivity is always positive [8]. Consistency requires $\alpha_n \ll \Omega_n$, which implies $\Omega_n \ll X_c = O(\ln L)$ as stated in Lemma 2.

Note that, for fixed $\delta X$, the separation betweeen the real parts of two consecutive eigenvalues decays to zero faster than $1/(\ln L)^2$:

$$\alpha_{2m+1} - \alpha_{2m-1} \sim \frac{8m\pi^2 J^3\,(f_2 - \frac{f_1^2}{2})}{X_c^3} = m\,O\left(\frac{1}{(\ln L)^3}\right) \ll \frac{1}{(\ln L)^2}. \tag{2.31}$$

From the dispersion relation (2.24), and given (2.31), we find the number of modes that become unstable when $\delta\phi$ is as in (2.22) (cf. (2.24) with $\alpha_N = 0$):

$$N = O(\sqrt{\delta\phi}\,\ln L), \tag{2.32}$$

which was considered when we wrote the values taken by $n$ in (2.23). For voltages $\phi = \phi_\alpha + \delta\phi$ or $\phi = \phi_\omega - \delta\phi$, with $\delta\phi$ in the range (2.22), a quasi continuum of eigenvalues with vanishing frequencies (2.23) crosses the imaginary axis. We shall present the analysis of the resulting bifurcation in § 5.

[1] There is an authentic spatial scale associated with $\chi$ because the field disturbance is a wave traveling with speed $J$ on the interval $0 < x < X_c + \Delta x$. See the interpretation of the amplitude equation in § 5.

*Remark* 4. The asymptotic description of the large-amplitude oscillations of the current due to solitary wave dynamics implies that these exist for $\phi > \phi_{crit} \sim E_1(J_c)\,L$ [22]. This conclusion was based upon the following: (a) the solitary wave leaves $E = E_1(J)$ behind once it has abandoned the neighborhood of $x = 0$ and reached maturity; (b) the excess area near $x = 0$ due to the gradual decay of $E(x; J)$ from $\rho J$ to $E_1(J)$ is neglected; and (c) the mechanism for solitary wave creation comes to a halt if $\phi$ is so small that the solitary wave disappears at $x = L$ for $J < J_c$. Clearly, it is hard to decide on this basis whether $\phi_\alpha > \phi_{crit}$ or $\phi_\alpha < \phi_{crit}$. In fact both situations might be possible for different values of the resistivity $\rho$. Elucidation of this point requires extensive numerical calculations which will not be attempted here.

**3. Linear inhomogeneous problem and secular terms.** We plan to perform a perturbation calculation of the oscillatory branch that bifurcates from the steady state both for finite $L$ and for $L \to \infty$. Then it is important to know under what conditions the solution of the linear nonhomogeneous problem associated with eqs. (2.13)–(2.15) (at the critical voltage) is bounded and periodic in time. In other words, we want to find the precise conditions under which no secular terms are present in our calculation. In this section we consider the simpler case of finite $L$ where only two complex conjugate zeros of the impedance (2.19), $\lambda = \pm i\,\omega$, cross the imaginary axis at the critical voltage. The linear nonhomogeneous problem is

$$\begin{gathered} \mathcal{L}\tilde{e} - \tilde{j} = f(x)\,e^{i\,\omega\,t}, \\ \tilde{e}(0,t) + \rho\left[\frac{\partial \tilde{e}(0,t)}{\partial t} - \tilde{j}(t)\right] = g\,e^{i\,\omega\,t}, \\ \int_0^L \tilde{e}(x,t)\,dx = 0. \end{gathered} \tag{3.1}$$

In (3.1), the operator $\mathcal{L}$ is given by (2.16), and the coefficients are calculated for $E = E(x; J)$, the steady state at the critical voltage. We now find the conditions that ensure absence of secular terms. Insertion of the ansatz

$$\tilde{j}(t) = \hat{j}\,e^{i\,\omega\,t}, \qquad \tilde{e}(x,t) = \hat{e}(x)\,e^{i\,\omega\,t}, \tag{3.2}$$

in (3.1) yields

$$\begin{gathered} \frac{\partial[v(E)\hat{e}]}{\partial x} + [i\omega + v'(E)]\hat{e} = \hat{j} + f(x), \\ \hat{e}(0) = \frac{\rho\,\hat{j} + g}{1 + i\rho\,\omega}, \\ \int_0^L \hat{e}(x)\,dx = 0. \end{gathered} \tag{3.3}$$

We now solve the first two equations and insert the solution in the third. The result is that a sum of three terms is zero. The first term, proportional to $\hat{j}$, is zero because it is the solution of the homogeneous problem (3.1) with $f \equiv 0$ and $g \equiv 0$. Then the sum of the remaining terms must also be zero for a bounded periodic solution of the type (3.2) to be possible:

$$\begin{aligned} \int_0^L \frac{dx}{v(E(x))} \Bigg\{ & \frac{g\,v(\rho J)}{1 + i\rho\omega} \exp\left[-\int_0^x \frac{i\omega + v'(E(z))}{v(E(z))}\,dz\right] \\ & + \int_0^x f(y) \exp\left[-\int_y^x \frac{i\omega + v'(E(z))}{v(E(z))}\,dz\right] dy \Bigg\} = 0. \end{aligned} \tag{3.4}$$

If the left side of (3.4) is not zero, the solution of (3.1) has to be proportional to $t\,e^{i\omega t}$, thereby yielding secular terms (unbounded as $t \to \infty$) in the perturbation theory of which the linear problem (3.1) is part. We will call (3.4) the nonresonance condition, and we will use it extensively in the bifurcation calculations that follow.

**4. Hopf bifurcation.** Let $E = E(x;\phi)$ be the steady state corresponding to a voltage $\phi$, and $J = J(\phi)$ the corresponding current. Let $E_0(x)$ and $J_c$ be the corresponding field and current at the critical voltage $\phi_c$ where the steady state ceases to be linearly stable. ($\phi_c$ is either $\phi_\alpha$ or $\phi_\omega$ of § 2.) We want to construct the time-periodic solutions that bifurcate from $E(x)$ at $\phi = \phi_c$. Let us define the small parameter $\epsilon$ as the deviation from the critical voltage $\phi_c$:

$$\phi = \phi_c + \epsilon^2\,\varphi, \quad \epsilon \ll 1. \tag{4.1}$$

(The sign of $\varphi = \pm 1$ will be determined later.) Then the corresponding stationary field and current are

$$\begin{aligned} E(x;\epsilon) &= E_0(x) + \epsilon^2\,\varphi\,E_2(x) + O(\epsilon^4),\\ J &= J_0 + \epsilon^2\,\varphi\,J_2 + O(\epsilon^4). \end{aligned} \tag{4.2}$$

Clearly, $E_2(x) = \partial E(x;\phi_c)/\partial\phi$, $J_2 = dJ(\phi_c)/d\phi$. To calculate the bifurcating oscillatory solution, we shall assume the usual Hopf multiscale ansatz [30], [9]:

$$\begin{aligned} E(x,t;\epsilon) &= E_0(x) + \epsilon\,E^{(1)}(x,t,T) + \epsilon^2[E^{(2)}(x,t,T) + \varphi E_2(x)]\\ &\qquad + \epsilon^3 E^{(3)}(x,t,T) + O(\epsilon^4),\\ J(t;\epsilon) &= J_0 + \epsilon\,J^{(1)}(t,T) + \epsilon^2[J^{(2)}(t,T) + \varphi J_2] + \epsilon^3 J^{(3)}(t,T) + O(\epsilon^4);\\ &\qquad t = t,\ T = \epsilon^2 t. \end{aligned} \tag{4.3}$$

The slow time scale $T = \epsilon^2 t$ is chosen so as to eliminate the secular terms that first appear in the equation for $E^{(3)}$ (see below). Inserting (4.3) into Equations (1.8), (1.9), and (1.4) and equating like powers of $\epsilon$, we obtain the following hierarchy of equations for the $E^{(k)}$'s and $J^{(k)}$'s:

$$\mathcal{L}_c E^{(1)} - J^{(1)} = 0, \tag{4.4}$$

$$E^{(1)}(0,t,T) + \rho\left[\frac{\partial E^{(1)}(0,t,T)}{\partial t} - J^{(1)}(t,T)\right] = 0, \tag{4.5}$$

$$\int_0^L E^{(1)}(x,t,T)\,dx = 0, \tag{4.6}$$

$$\mathcal{L}_c E^{(2)} - J^{(2)} = -\frac{1}{2}\left[\frac{\partial}{\partial x}v' + v''\right](E^{(1)})^2, \tag{4.7}$$

$$E^{(2)}(0,t,T) + \rho\left[\frac{\partial E^{(2)}(0,t,T)}{\partial t} - J^{(2)}(t,T)\right] = 0, \tag{4.8}$$

$$\int_0^L E^{(2)}(x,t,T)\,dx = 0, \tag{4.9}$$

$$\mathcal{L}_c E^{(3)} - J^{(3)} = -\left[\frac{\partial}{\partial T} + \varphi\left(\frac{\partial}{\partial x} v'\, E_2 + v''\, E_2\right)\right] E^{(1)} \tag{4.10}$$
$$-\left[\frac{\partial}{\partial x}\, v' + v''\right] E^{(1)} E^{(2)} - \frac{1}{6}\left[\frac{\partial}{\partial x}\, v'' + v'''\right] (E^{(1)})^3,$$

$$E^{(3)}(0,t,T) + \rho\left[\frac{\partial E^{(3)}(0,t,T)}{\partial t} - J^{(3)}(t,T)\right] = -\rho\, \frac{\partial E^{(1)}}{\partial T}, \tag{4.11}$$

$$\int_0^L E^{(3)}(x,t,T)\, dx = 0. \tag{4.12}$$

Here $\mathcal{L}_c$ is the operator (2.16) evaluated at $\epsilon = 0$. The argument of the function $v$ and of its derivatives here and in what follows is thus $E = E_0(\cdot)$. Terms such as $[\frac{\partial}{\partial x}\, v' + v''] \, (E^{(1)})^2$ mean $\frac{\partial [v' \, (E^{(1)})^2]}{\partial x} + v'' \, (E^{(1)})^2$, so that the operator $[\frac{\partial}{\partial x}\, v' + v'']$ acts on whatever function of $x$ follows it.

The solution of Eqs. (4.4)–(4.6) is

$$J^{(1)}(t,T) = A(T) e^{i\,\omega\, t} + cc,$$
$$E^{(1)}(x,t,T) = A(T)\, e^{i\,\omega\, t}\, \psi(x) + cc, \tag{4.13}$$

where $\psi(x)$ is the solution of eqs. (2.18)–(2.20) corresponding to the eigenvalue $\lambda = i\omega$ and $cc$ means the complex conjugate of the preceding term. We now insert (4.13) in (4.7) and solve the resulting linear nonhomogeneous equation with the boundary condition (4.8). We find

$$J^{(2)}(t,T) = \nu_0 \mid A(T) \mid^2 + \nu_2\, A(T)^2\, e^{i\, 2\omega\, t} + cc,$$
$$E^{(2)}(x,t,T) = \xi_0(x) \mid A(T) \mid^2 + \xi_2(x)\, A(T)^2 e^{i\, 2\omega\, t} + cc, \tag{4.14}$$

where

$$\xi_n(x)\, v(E_0(x)) = \nu_n \left[\frac{\rho\, \Theta_n(0,x)}{1 + i\, n\, \rho\, \omega} + \int_0^x \Theta_n(y,x)\, dy\right] - \frac{1+\delta_{n0}}{2} \tag{4.15}$$
$$\int_0^x dy\, \Theta_n(y,x) \left[\frac{\partial}{\partial y}\, v' + v''\right] \left[\psi(y)^2 \left(\frac{\bar{\psi}(y)}{\psi(y)}\right)^{\delta_{n0}}\right],$$

$$\nu_n = \frac{(1+\delta_{n0}) \int_0^L \frac{dx}{v} \int_0^x dy\, \Theta_n(y,x) [\frac{\partial}{\partial y} v' + v''] \, [\psi(y)^2 \frac{\bar{\psi}(y)}{\psi(y)})^{\delta_{n0}}]}{2 \int_0^L \frac{dx}{v} [\frac{\rho \Theta_n(0,x)}{1+in\rho\omega} + \int_0^x \Theta_n(y,x)\, dy]}, \tag{4.16}$$

$$\Theta_n(y,x) \equiv \exp\left[-\int_y^x \frac{v'(E_0(z)) + i\, n\, \omega}{v(E_0(z))}\, dz\right], \tag{4.17}$$

where $n$ is an integer ($n = 0, 2$ in (4.15)–(4.16)). In (4.14), we have omitted terms that decay exponentially to zero in the fast time scale $t$. $\delta_{n0}$ is the Kronecker delta, equal to 1 for $n = 0$ and to zero otherwise.

To find the equations for the amplitude $A(T)$, we insert (4.13) and (4.14) in the right side of (4.10). Then for the solution of this problem to be bounded and time periodic in the fast time scale $t$, we need to impose the nonresonance condition (3.4). The result is the following amplitude equation for $A(T)$:

$$\frac{dA}{dT} = \varphi\,\lambda_1\,A - \gamma\,A\ \mid A\mid^2 \tag{4.18}$$

with

$$\lambda_1 = -\frac{1}{\mathcal{D}}\int_0^L \frac{dx}{v}\int_0^x dy\,\Theta_1(y,x)\left[\frac{\partial}{\partial y}v' + v''\right](\psi\,E_2), \tag{4.19}$$

$$\begin{aligned}\gamma = \frac{1}{\mathcal{D}}\int_0^L \frac{dx}{v}\int_0^x dy\,\Theta_1(y,x)\Bigg\{&\left[\frac{\partial}{\partial y}v' + v''\right](\xi_0\psi + \xi_2\bar{\psi})\\ &+\frac{1}{2}\left[\frac{\partial}{\partial y}v'' + v'''\right]\psi\mid\psi\mid^2\Bigg\},\end{aligned} \tag{4.20}$$

and

$$\mathcal{D} = \int_0^L \frac{dx}{v}\left\{\left[\frac{\rho}{1+i\,\rho\,\omega}\right]^2 v(\rho J)\,\Theta_1(0,x) + \int_0^x dy\,\Theta_1(y,x)\,\psi(y)\right\}. \tag{4.21}$$

Note that $\lambda_1 = \frac{\partial\lambda}{\partial\phi}$ at $\phi = \phi_c$ (Appendix B). Thus Re$\lambda_1 > 0$ at the lower critical voltage $\phi_\alpha$ and Re$\lambda_1 < 0$ at the upper critical voltage $\phi_\omega$.

Eq. (4.18) has the following periodic solution:

$$\begin{aligned}A(T) &= \mathrm{Re}^{i\,\varphi\,\mu\,(T-T_0)},\ T_0 = \text{const.},\\ R &= \sqrt{\frac{\varphi\,\mathrm{Re}\,\lambda_1}{\mathrm{Re}\,\gamma}},\\ \mu &= \mathrm{Im}\,\lambda_1 - \frac{\mathrm{Im}\,\gamma\,\mathrm{Re}\,\lambda_1}{\mathrm{Re}\,\gamma}.\end{aligned} \tag{4.22}$$

We now show that the stability of the solution (4.22) depends on the sign of Re$\gamma$ only. Let Re$\gamma > 0$. Then the bifurcating solution is asymptotically stable (except for the constant phase shift $T_0$). In fact, if Re$\lambda_1 > 0$, the solution (4.22) exists for $\varphi = 1$ (therefore $\phi > \phi_c = \phi_\alpha$) and it is reached for large positive times starting from any initial condition different from $A = 0$. If Re$\lambda_1 < 0$, the solution (4.22) exists for $\varphi = -1$ (therefore $\phi < \phi_c = \phi_\omega$), and it is again reached for large positive times starting from any initial condition different from $A = 0$. Similarly, we can show that the bifurcating solution is unstable when Re$\gamma < 0$. This is the usual "principle of exchange of stabilities" [30], [9] between the trivial stationary solution and the bifurcating branch of oscillatory solutions: *a supercritical solution (bifurcating toward the side where the steady state is unstable) is stable, whereas a subcritical solution (bifurcating toward the side where the steady state is stable) is unstable.*

We have numerically evaluated the coefficient Re$\gamma$ for $\phi = \phi_\alpha$ and $L$ near the minimum length at $\rho = 1.8$ [28]. The result is that (i) Re$\gamma < 0$ (subcritical bifurcation) near $L = L_m$; (ii) for intermediate lengths, $L \in (L_1, L_2)$, Re$\gamma > 0$ (supercritical bifurcation); (iii) whereas for $L > L_2$, again Re$\gamma < 0$ (subcritical bifurcation, which agrees with the asymptotic result for $\ln L \gg 1$ obtained in the next section). At $L_1$ and $L_2$ there are transitions from subcritical to supercritical and from supercritical back to subcritical Hopf bifurcations. Discussions on the physical implications of these results may be found in § 6.

**5. Amplitude equation for ln $L \gg 1$.** For very long semiconductors, Lemma 2 of § 2 indicates that many modes (2.32) become unstable shortly after the voltage crosses its critical value, (2.22). A convenient representation of the electric field and the current on the bifurcating oscillatory branch for $\ln L \gg 1$ is given by

$$E(x,t,T;\epsilon) - E_0(x) = \epsilon \sum_{n\,\mathrm{odd}} A_n(T)\, e^{i\Omega_n t}\, \psi_n(x) + O(\epsilon^2),$$

$$J(t,T;\epsilon) - J_0 = \epsilon \sum_{n\,\mathrm{odd}} A_n(T)\, e^{i\Omega_n t}. \tag{5.1}$$

Here $\psi_n(x) = \hat{e}(x; i\,\Omega_n)/\hat{j}$ as in Appendix A, with $\Omega_n$ given by (2.23). The sums are over all odd integers, and they include both unstable modes with $n$ of the order of $N$ given by (2.32), as well as linearly stable modes with larger $n$. Since the solutions have to be real, $A_{-n} = \bar{A}_n$ in (5.1) and in what follows. Equations (5.1) are obtained by solving equations (4.4)–(4.5) for $E^{(1)}$ and $J^{(1)}$, with exclussion of linearly stable modes corresponding to eigenvalues whose real part is $O(1)$.

The linear parts of the amplitude equations for $A_n$ would determine their evolution if all the $A_n$ were small. They follow straightforwardly from the dispersion relation (2.24):

$$\frac{\partial A_n}{\partial T} = \frac{|v_2'|\varphi}{2\,X_c\,(E_2 - E_1)}\,A_n - \frac{J\,(f_2 - (f_1^2/2))}{X_c}\left(\frac{nJ\pi}{\epsilon X_c}\right)^2 A_n, \tag{5.2}$$

where $n = \pm 1, \pm 3, \ldots, O(\epsilon X_c)$. This will be derived by means of the multiscale method of § 4 in Appendix C.

The nonlinear terms of the amplitude equation, missing in (5.2), couple different $A_n$'s and are essential when these are of order 1. They will be derived in Appendix C, but their form can be advanced from the following considerations:

1. The nonlinear terms are cubic in the $A_n$'s. The reason is the same as in § 4: resonant terms first appear in the equations for $E^{(3)}$ that contain products of three factors of the form (5.1).
2. The coefficient of each trinomial $A_p A_q A_r$ appearing in the amplitude equation for $A_n$ is independent of $(p, q, r, n)$. This is a consequence of the limit $\ln L \to \infty$: $\psi_n(x) \sim \hat{e}(x; 0)/\hat{j}$, as $n = O(\epsilon X_c)$, which implies $\Omega_n \ll 1$. We show in Appendix C that, as far as a determination of the leading order of the nonlinear term in the amplitude equations goes, the integer $n$ does not appear in the right-hand sides of the equations for $E^{(2)}$ and $E^{(3)}$. Then the nonlinear term in the amplitude equation for $A_n$ has the form

$$-\gamma_\infty \sum_{p,q\ \mathrm{odd}} A_p A_q A_{n-p-q}. \tag{5.3}$$

3. The coefficient of the sum (5.3) is $\gamma_\infty = (3X_c)^{-1} \lim_{L\to\infty}(X_c\gamma)$, where $\gamma$ is given by (4.20). (See Appendix C for an alternative direct derivation and an explicit evaluation of $\gamma_\infty$.) This result follows from the complete amplitude equation with initial condition $A_n(0) = \bar{A}_{-n}(0) = \delta_{n1}$. Clearly, $A_1 = \bar{A}_{-1}$ is the only excited mode for $T > 0$, and its evolution should obey (4.18). The factor 3 in $\gamma_\infty$ represents the different ways of obtaining $A_1\,|A_1|^2$ from the sum in (5.3).

Putting together (5.2) and (5.3), we obtain the leading-order approximation to the amplitude equation in the limit $\ln L \to \infty$:

$$X_c \frac{\partial A_n}{\partial T} = (\alpha\varphi - \beta\Omega_n^2)\, A_n - \Gamma\, L^2 \sum_{p,m\,\mathrm{odd}} A_p\, A_m\, A_{n-p-m}, \tag{5.4}$$

where the coefficients

$$\alpha = \frac{|\, v_2' \,|}{2\,(E_2 - E_1)} > 0,$$

$$\beta = J\left(f_2 - \frac{f_1^2}{2}\right) > 0,$$

$$\Gamma = -\frac{v\prime_2^2}{12\, J\, v\prime_1^2\, (E_2 - E_1)^2} \sim \frac{\gamma_\infty}{L^2} < 0, \tag{5.5}$$

are independent of $n$ (cf. Appendix C for the derivation of the explicit expression of $\Gamma$ in (5.5)).

We can rewrite (5.4) as the following equation for

$$u(\chi, \tau) = \frac{J^{(1)}(t,T)}{L},\ \chi = \epsilon t,\ \tau = \frac{\epsilon^2 t}{X_c}: \tag{5.6}$$

$$\frac{\partial u}{\partial \tau} = \beta\, \frac{\partial^2 u}{\partial \chi^2} + (\alpha\varphi - \Gamma\, u^2)\, u; \tag{5.7}$$

$$u\left(\chi + \frac{X_c\, \epsilon}{J}, \tau\right) = -u(\chi, \tau). \tag{5.8}$$

The antiperiodicity condition (5.8) results from (5.1) and (2.21): note that (5.7) is a reaction-diffusion equation where both "time" $\tau$ and "space" $\chi$ are slow time scales. Our result $\Gamma < 0$ indicates that $\varphi = -1$ and the bifurcation is subcritical. Then any solution of (5.7)–(5.8) blows up in finite "time" $\tau$. This means that our perturbation scheme breaks down, and presumably strong derivatives or discontinuities in the current (and consequently in the field) are created. These may then evolve into solitary waves with a back shock or into traveling monopole fronts like those discussed in [22].

Had a result $\Gamma > 0$ been found, a different situation would have occurred. In this case $\varphi = 1$ and the stable supercritical solution of (5.7)–(5.8) would have been a $\tau$-independent steady state of large period $2\,\epsilon\, X_c/J \gg 1$. In the phase plane $(u, \frac{\partial u}{\partial \chi})$ such a solution corresponds to a closed orbit that (i) spends a long time $\chi$ near the saddle points $(\pm\sqrt{\alpha/\Gamma}, 0)$ and (ii) jumps from one saddle to the other along the heteroclinic orbits connecting them. One period of $u(\chi)$ is approximately a front connecting $(-\sqrt{\alpha/\Gamma}, 0)$ and $(\sqrt{\alpha/\Gamma}, 0)$ followed after $\chi \sim \epsilon X_c/J$ by the "opposite" front connecting back $(\sqrt{\alpha/\Gamma}, 0)$ and $(-\sqrt{\alpha/\Gamma}, 0)$. In the original variables, we have

$$E(x,t;\epsilon) - E_0(x) \sim \frac{\epsilon}{L} \sum_{n\,\mathrm{odd}} \hat{u}_n \frac{c_L\, v_2'\, L\, e^{|v_2'|\,(x - X_c)/J}}{J\, v_1'(E_2 - E_1)}\ \exp\left[-\frac{i\, n\pi(x - X_c - Jt)}{X_c}\right] \tag{5.9}$$

for $0 < x < X_c + \Delta x$. Here $\hat{u}_n$ is the Fourier coefficient of the antiperiodic solution of (5.7)–(5.8). $E(x,t;\epsilon) - E_0(x)$ decreases exponentially to zero as $(x - X_c - \Delta x)$ grows. One period of $E(x,t;\epsilon)$ may thus be described as follows. The front connecting the two saddle points mentioned above moves away from $x = 0$ with speed $J$ and an amplitude that grows exponentially with $x$. After reaching $x \sim X_c$, two things happen: (i) the front enters $(X_c + \Delta x, L]$ and is exponentially attenuated as it moves further; (ii) at $x = 0$ another front with slope of different sign to the first one is created and starts moving. When this second front reaches $x \sim X_c$, a new front like the first one is created at $x = 0$ and the period is completed. This situation is akin to a Gunn effect mediated by waves that decay after penetrating a distance $x \sim X_c$ in the semiconductor. In [18] such a situation was numerically observed for a curve $v(E)$ which saturated after the maximum at $E = E_M$ was reached, when the boundary condition at $x = 0$ gave rise to a field on the saturating region[2] of $v(E)$ (cf. also [42, § 4-4, particularly Figs. 4.3 and 4.4]). A similar behavior has also been observed in another semiconductor model [13]. For this model the reduced equation is also hyperbolic but now second order, and additional terms give contributions to $\Gamma$ in (5.7) that may account for a supercritical bifurcation without invoking subdominant terms. This confinement of the wave dynamics to a small part of the semiconductor is provoked by the boundary condition we have used and is related to similar phenomena in other pattern-forming systems [14]. Experimental confirmation in p-Ge may be found in [24].

*Remark* 5. It is important to note that the amplitude of the oscillations grows as $|\phi - \phi_\alpha|^{1/2}/L$ according to (5.6) and (5.9). Thus the interval of voltages for which our asymptotic expansion holds increases with $L^2$, and therefore the corresponding interval of "average" electric fields $(\phi/L)$ increases with $L$. In physical words, the onset region where bistability or intermittency may be expected increases with semiconductor length. This agrees qualitatively with experimental observations in p-Ge (for which similar analyses of the instability have been performed [13]), where phenomena in the onset region were reported only for quite large samples [25].

**6. Discussion.** We have performed the calculation of the Hopf bifurcation of small-amplitude current oscillations at the critical voltage for the standard model of the Gunn instability in n-GaAs. Our method for dealing with a partial differential equation coupled with an integral conservation law is clearly applicable to problems in other fields. For example, to the problem of synchronization of infinitely many oscillators coupled with their mean field, in the presence of frequency noise and white noise [9], the one-oscillator probability distribution obeys a drift-diffusion equation (Fokker–Planck) subjected to an integral constraint which establishes the link between the order parameter and the probability.

Let us return to the semiconductor model for n-GaAs. In the previous sections we have shown that the oscillatory branch bifurcating at $\phi_\alpha$ may be subcritical or supercritical, according to the value of $L$. We have not explored in detail what may happen at $\phi_\omega$, the end of the Gunn instability, except for $L$ near $L_m$ and for $B = O(1)$ as $\ln L \to \infty$, where the Hopf bifurcation is subcritical (cf. the end of § 4 and also (5.5)). For large $L$ and small $B$ the analysis of § 4 may not be applicable because $v_1' \sim B \ll 1$ (cf. § 5). Similarly, our study of the steady state and its instability does not apply to the more realistic case of a saturating $v(E)$ curve near $\phi_\omega$. See [42] for numerical simulations of this case.

[2] This provides a small enough $|v_2'|$, and therefore the subdominant terms that we have neglected in our derivation might be of the same order as those kept here. A supercritical situation would then be a possible outcome, which would explain the phenomena observed by [18].

Putting together the analysis of the onset and fully developed Gunn instability provides us with a global bifurcation diagram and poses several new problems. Let us consider the case of a long semiconductor $L \gg 1$ where asymptotic analyses of both the onset and fully developed Gunn instability [22] exist. (For $L = O(1)$, we have a Hopf bifurcation at simple eigenvalues. The changes from subcritical to supercritical bifurcation and then back to subcritical bifurcation as $L$ increases may be interpreted with the help of the bifurcation diagrams in [16].) Very briefly, the voltages at which the steady state become unstable, $\phi_\alpha$ given by (2.6) and (2.21), and at which the Gunn oscillations appear, $\phi_c$ [22], are of the same order, $E_1(J_c)\, L$. There are two possibilities:

1. $\phi_\alpha > \phi_c$. There is bistability between the stable steady state and the Gunn oscillations for $\phi_c < \phi < \phi_\alpha$ for the case of (5.5), in which the Hopf bifurcation at the onset of the instability is subcritical.
2. $\phi_\alpha < \phi_c$. For the case of (5.5) (subcritical bifurcation), there is no stable solution in the voltage interval $\phi_\alpha < \phi < \phi_c$. Given that, once a solitary wave has been created, it moves to $x = L$ and that the instability of the steady state is concentrated near $x = 0$, a possible outcome is a type 1 intermittency [35]: there would be "laminar" stages where $J(t)$ would take its steady value, followed by irregular firings of solitary waves near $x = 0$. This phenomenon might have been observed in n-GaAs by Kabashima, Yamazaki, and Kawakubo in 1976 [23] (especially pp. 923–924): they also reported (very small) hysteretic phenomena between oscillations and the steady state. See also Gunn's experiments on long n-GaAs samples [20]. For the supercritical bifurcation case (conjectured for a saturating $v(E)$ and large contact resistivity), it could be that the bifurcating solution branch smoothly becomes the Gunn oscillation branch or that these branches are disjoint. If the latter is the case, we could have either bistability between small- and large-amplitude current oscillations or an intermittency[3] region where there are no stable time-periodic or stationary solutions.

Further numerical and asymptotic analyses [28] are needed to ascertain which possibilities are realized in the present or related models. One open problem is the role played by fluctuations in the origin of the experimentally observed intermittency at the onset of Gunn-type instabilities [23], [25]. Fluctuations have been modelled by stochastic equations, either in mode-mode coupling theories [39] or in fluctuating hydrodynamic models [27], [15]. In all these works the fluctuations are considered small disturbances about the steady state for $\phi \leq \phi_\alpha$ which may not capture the intrinsically nonlinear solitary wave dynamics essential to the Gunn instability.

Interesting open problems are posed by the case of alternating current (AC)+DC voltage bias. Numerical solution of drift-diffusion models show temporal chaos with wave dynamics of the electric field [38]. Experiments in p-Ge [26] demonstrate both this type of temporal chaos and spatiotemporal chaos where the wave dynamics is such that different parts of the semiconductor are uncorrelated. Except for some preliminary numerical calculations, there is little analysis of the relevant model [3] with AC+DC voltage bias. Nothing is known about the possible effect of fluctuations.

Finally, let us point out that the drift-diffusion model we have studied here is related to the continuum limit of discrete drift models of electric field domains in doped superlattices [5]. A Gunn effect due to periodic formation, motion, and annihilation of

[3] Now a type 3 intermittency between small, and large-amplitude current oscillations, of the type observed by Kahn et al in p-Ge [25].

domain walls (charge monopoles) seems to be responsible for some of the time-periodic oscillations of the photocurrent experimentally observed [33].

**Appendix A. Linear stability of the steady state for $\ln L \gg 1$.** We recall here the more salient features of the proof of Lemma 2 [8]. As said in Lemma 1, the steady state differs appreciably from the piecewise constant profile

$$E = E_2,\ 0 < x < X, \qquad E = E_1,\ X < x < L, \tag{A.1}$$

only in a transition layer of $O(1)$ width centered at $x = X$ and in a narrower boundary layer of $O(\delta)$ width at $x = L$. Near the critical voltages $\phi_\alpha$ and $\phi_\omega$, we have $1 \ll X \ll L$. It is convenient to split the solution of (2.18)–(2.20) as follows:

$$\hat{e}(x;\lambda) = \hat{e}_p(x;\lambda) + h(x;\lambda)\,\frac{\partial E(x;J)}{\partial x}, \tag{A.2}$$

where $\hat{e}_p(x;\lambda)$ obeys (2.18) with the natural boundary condition

$$\hat{e}_p(0;\lambda) = \frac{\hat{\jmath}}{v_2' + \lambda} \sim \hat{e}_p(X - \Delta x;\lambda)(\text{as}\,\Delta x \to \infty). \tag{A.3}$$

Any solution of (2.18), $\hat{e}_p(x;\lambda)$ in particular, satisfies

$$\hat{e}_p(X + \Delta x;\lambda) \sim \frac{\hat{\jmath}}{v_1' + \lambda}\ (\text{as}\,\Delta x \to \infty). \tag{A.4}$$

The second term in (A.2) solves (2.18) with $\hat{\jmath} = 0$ and an appropriately modified boundary condition. It is given by [8],

$$h(x;\lambda) = h(X;\lambda)\,\exp\!\left[-\lambda\,\frac{x - X + E(x;J) - E_0}{J}\right],$$
$$h(X;\lambda) \sim -\frac{(1-\rho v_2')\,J\hat{\jmath}}{(1+\rho\lambda)\,(v_2'+\lambda)\,c_L\,v_2'}\,\exp\!\left[-\frac{(v_2'+\lambda)\,X + (E_0 - E_2)\lambda}{J}\right]. \tag{A.5}$$

Here $E_0$ is as defined in Lemma 1. As $(x - X) \to \infty$, the second term in (A.2) becomes exponentially small, so that $\hat{e} \sim \hat{e}_p$ in $(X + \Delta x, L]$ and

$$\int_{X+\Delta x}^{L} \hat{e}\,dx \sim \frac{(L - X - \Delta x)\hat{\jmath}}{v_1' + \lambda} \sim \frac{L\hat{\jmath}}{v_1' + \lambda}. \tag{A.6}$$

For $\lambda$ to be a zero of $Z(\lambda)$ in (2.19), the integral of $\hat{e}$ on $[0, X + \Delta x)$ has to be of the same order as (A.6). The contribution of $\hat{e}_p$ to this integral is, according to (A.3) and (A.4), $O(X)$, much smaller than needed. Thus it is the contribution coming from the second term of (A.2) that should balance (A.6). This yields

$$Z(\lambda) \sim \frac{L}{v_1' + \lambda} - \frac{(1-\rho v_2')\,J\,K(\lambda)}{(1+\rho\lambda)\,(v_2'+\lambda)\,c_L\,v_2'}\,\exp\!\left[-\frac{(v_2'+\lambda)\,X}{J}\right],$$
$$K(\lambda) = \int_0^{X+\Delta x} \frac{\partial E}{\partial x}\,\exp\!\left[-\lambda\,\frac{x - X + E(x;J) - E_2}{J}\right] dx$$
$$= (E_2 - E_1)\,(1 + \phi_1\,\lambda + \phi_2\,\lambda^2 + \cdots). \tag{A.7}$$

By means of (A.7), we can rewrite the impedance condition $Z(\lambda) = 0$ as follows:

$$\frac{(E_2 - E_1)\,(1 - \rho v_2')\,J\,v_1'}{c_L\,v_2'^2}\,F(\lambda)\,\exp\left\{-\frac{(v_2' + \lambda)\,X}{J} - \ln L\right\} \sim -1,$$

(A.8)
$$F(\lambda) = 1 + f_1\,\lambda + f_2\,\lambda^2 + \cdots,$$

where $f_1$ and $f_2$ are given by (2.26) and (2.27), respectively. A study of (A.8) yields the formulas in Lemma 2 for $X_c$, the critical frequencies, and the dispersion relation, [8].

**Appendix B. In the amplitude equation (4.18), $\lambda_1 = \frac{\partial\lambda}{\partial\phi}$ at $\phi = \phi_c$.** To prove this result, let us differentiate the equations (2.18)–(2.20) for $\hat{e}$ with respect to $\phi$. We obtain an equation for $\frac{\partial\hat{e}}{\partial\phi} \equiv \hat{e}_\phi$ which can be simplified by using the equations for $\frac{\partial E}{\partial\phi} \equiv E_\phi$ (found by differentiating the steady-state equations (2.1)–(2.3) with respect to $\phi$). The result is

(B.1)
$$\frac{\partial[v(E)\,\hat{e}_\phi]}{\partial x} + [\lambda + v'(E)]\,\hat{e}_\phi = -[\lambda_\phi + v''(E)\,E_\phi]\,\hat{e} - \frac{\partial[v'(E)\,E_\phi\,\hat{e}]}{\partial x},$$

(B.2)
$$\hat{e}_\phi(0) = -\frac{\rho\,\lambda_\phi}{1 + \rho\,\lambda}\,\hat{e}(0;\lambda),$$

(B.3)
$$\int_0^L \hat{e}_\phi(x)\,dx = 0.$$

Here $\frac{\partial\lambda}{\partial\phi} \equiv \lambda_\phi$. We see that $\lambda_\phi\,\hat{e}$ and $\hat{e}\,E_\phi$ play the same role in (B.1) and (B.2) as $\frac{\partial E^{(1)}}{\partial T}$ and $E_2\,E^{(1)}$ in (4.4) and (4.5), respectively. Using the nonresonance condition, that is, inserting the solution of (B.1) and (B.2) in (B.3), we should obtain for $\lambda_\phi$ the same expression (4.19) as found for the coefficient of the linear part of the amplitude equation (4.18).

**Appendix C. Direct derivation of the amplitude equation for $\ln L \gg 1$.** We adopt the usual ansatz (4.3). Inserting it into the equations and equating to zero the $O(\epsilon)$ terms, we obtain (4.4)–(4.6). For $\ln L \gg 1$, the solution of equations (4.4)–(4.5) is

$$E^{(1)}(x,t,T) = \sum_{n\,\mathrm{odd}} A_n(T)\,e^{i(\Omega_n+\omega_n)\,t}\,\psi_n(x),$$

(C.1)
$$J^{(1)}(t,\chi,T) = \sum_{n\,\mathrm{odd}} A_n(T)\,e^{i(\Omega_n+\omega_n)\,t}.$$

Here $\psi_n(x) = \hat{e}(x; i\Omega_n + i\omega_n)/\hat{j}$ as in Appendix A, with $\Omega_n$ given by (2.23) and $\omega_n$ given by (2.25). The sums are over all odd integers and they include both unstable modes with $n$ of the order of $N$ given by (2.32), as well as linearly stable modes with larger $n$. Since the solutions have to be real, $A_{-n} = \bar{A}_n$ in (C.1) and in what follows.

When we insert (C.1) in (4.6), we obtain

(C.2)
$$\int_0^L \psi_n(x)\,dx \sim \frac{(f_2 - \frac{f_1^2}{2})\,L}{v_1'}\,\Omega_n^2.$$

The right side on this expression is $O(\epsilon^2 L)$ because $\Omega_n = O(\sqrt{\delta X}) = O(\epsilon)$ from the dispersion relation. This means that, to the order in $\epsilon$ we are considering, (C.1) is the solution sought. Nevertheless, when using it, we need to add (C.2) times $\epsilon^{-2}$ to the left-hand side of (4.12) to obtain a correct result (see below).

We now insert (C.1) in (4.7) and (4.8) and solve these equations with the ansatz:

$$\begin{aligned} E^{(2)}(x,t,T) &= \sum_{p,q\, odd} e^{2i\Omega_n t}\, \delta_{p+q,2n}\, A_p(T)\, A_q(T)\, \xi_{p,2n}(x), \\ J^{(2)}(t,T) &= \sum_{p,q\, odd} e^{2i\Omega_n t}\, \delta_{p+q,2n}\, A_p(T)\, A_q(T)\, \nu_{p,2n}. \end{aligned} \tag{C.3}$$

We have ignored the correction to the frequency $\Omega_n$ as it will not affect the nonresonance condition for (4.10)–(4.12) which yields the amplitude equations to the order we consider here. Notice that if we restrict $p$ and $q$ to $\pm 1$ as in § 4, there are only four terms in these sums and $\xi_{1,0} = \xi_{-1,0}$ coincides with $\xi_0/2$ of (4.14), while $\xi_{1,2}$ equals $\xi_2$ of (4.14). Thus many terms are equal in the sums entering equation (C.3), but we will not count their multiplicity: we prefer to keep them all in what follows without gathering identical terms together. The constants $\nu$ are determined as we did in § 4, by imposing (4.9). We find the following equations for the $\xi$'s and the $\nu$'s:

$$\boldsymbol{L}\xi_{p,2n} \equiv \left(\frac{\partial}{\partial x}v + v' + i\,2\,\Omega_n\right)\xi_{p,2n} = \nu_{p,2n} - \frac{1}{2}\left(\frac{\partial}{\partial x}v' + v''\right)\psi_p\,\psi_{2n-p}, \tag{C.4}$$

$$\int_0^L \xi_{p,2n}(x)\, dx = 0. \tag{C.5}$$

We can approximate the solutions of these equations by exploiting the known structure of the steady state and of $\hat{e}$ for $\ln L \gg 1$ (cf. § 2, Appendix A, and, more fully, [8]). It is convenient to split $\xi_{p,2n}$ in three parts:

$$\xi(x) = \Xi^{(1)}(x) + \Xi^{(2)}(x) + h^{(2)}(x)\,\frac{\partial E}{\partial x}, \tag{C.6}$$

where

$$\begin{aligned} \boldsymbol{L}\Xi^{(1)} &= \nu, \\ \Xi^{(1)}(0) &= \frac{\nu}{v_2' + i\,2\,\Omega_n}, \end{aligned} \tag{C.7}$$

$$\begin{aligned} \boldsymbol{L}\Xi^{(2)} &= -\frac{1}{2}\left(\frac{\partial}{\partial x}v' + v''\right)\psi_p\,\psi_{2n-p}, \\ \Xi^{(2)}(0) &= 0, \end{aligned} \tag{C.8}$$

$$\begin{aligned} v(E)\,\frac{\partial h^{(2)}}{\partial x} + i\,2\,\Omega_n\, h^{(2)} &= 0, \\ h^{(2)}(0)\,\tfrac{\partial E(0;J)}{\partial x} &= -\frac{\nu\,(1-\rho\, v_2')}{(1+\rho\, i\,2\,\Omega_n)\,(v_2' + i\,2\,\Omega_n)}. \end{aligned} \tag{C.9}$$

Here we have suppressed the subscripts in $\xi_{p,2n}$, $\nu_{p,2n}$, and in $E_0$, as it will turn out that functions with different subscripts are equal to leading order in the limit $\ln L \to \infty$ in which $\Omega_n \ll 1$. First of all, recall that $\psi_n$ is $O(L)$ on $[0, X_c + \Delta x)$ and $O(1)$ on $(X_c + \Delta x, L]$ (with $1 \ll \Delta x \ll X_c$) according to (2.29), and (2.30), respectively. Using these orders of magnitude, it is straightforward to show that on $[0, X_c + \Delta x)$, the solutions of (C.7), (C.8), and (C.9) are of order $\nu$, $L^2$, and $\nu L$, respectively, whereas they are $O(\nu)$, $O(1)$, and exponentially small on $(X_c + \Delta x, L]$.

The largest contributions to the integral (C.5) are thus those of $\Xi^{(2)}$ and $h^{(2)}\,\frac{\partial E}{\partial x}$ on $[0, X_c + \Delta x)$ and of $\Xi^{(1)}$ on $(X_c + \Delta x, L]$. Keeping them, (C.5) yields

$$\text{(C.10)} \qquad \frac{2\nu L}{v_1'} \sim \frac{L^2\, v_2'}{2\, v_1'^2\, J\, (E_2 - E_1)} \Longrightarrow \nu \sim \frac{L\, v_2'}{4\, v_1'\, J\, (E_2 - E_1)}.$$

We find

$$\xi(x) \sim \frac{L^2}{4\, v_1'^2\, (E_2 - E_1)^2} \left( \frac{v_2'}{J} - \frac{2\, J\, v'}{v^2} \right) \frac{\partial E}{\partial x}, \; 0 < x < X_c + \Delta x,$$

$$\text{(C.11)} \qquad \xi(x) \sim \frac{L^2\, v_2'}{4\, v_1'^2\, J\, (E_2 - E_1)}, \; X_c + \Delta x < x < L.$$

A similar discussion shows that $J_2 = O(1/L)$ and that the stationary solution $E_2$ can be neglected in comparison to $E^{(2)}$ in (4.10) when deriving the linear part of the amplitude equations for the $A_n$'s, which we will do later. We now calculate the nonlinear part of the amplitude equation. The largest relevant terms on the right side of (4.10) are to leading order sums of $A_m A_p A_{n-m-p}$ over all odd $p, m$ times the following common function:

$$\text{(C.12)} \qquad \frac{L^3\, (\partial E/\partial x)}{6\, v_1'^3\, (E_2 - E_1)^3} \left[ J\, \frac{v''' - 3v'v''}{v^2} \left( \frac{\partial E}{\partial x} \right)^3 + \frac{3}{2} \left( \frac{Jv''}{v^2} + v' \frac{\partial}{\partial x} \right) \left( \frac{v_2'}{J} - \frac{2Jv'}{v^2} \right) \left( \frac{\partial E}{\partial x} \right)^2 \right] \text{ on } 0 < x < X_c + \Delta x,$$

$$\text{(C.13)} \qquad - \frac{L\, v_1''\, v_2'}{4\, v_1'^3\, J\, (E_2 - E_1)} \quad \text{on } X_c + \Delta x < x < L.$$

After some simplification, (C.12) can be rewritten as $v\, \frac{\partial \eta}{\partial x}\, \frac{\partial E}{\partial x}$, where

$$\text{(C.14)} \qquad \eta \sim \frac{L^3}{12\, v_1'^3\, (E_2 - E_1)^3} \left\{ \frac{v_2'}{J} \left( \frac{v_2'}{J} + 3\, \frac{\partial (J/v)}{\partial E} \right) + \frac{\partial^2}{\partial E^2} \left( \frac{J}{v} - 1 \right)^2 \right\}$$

on $0 < x < X_c + \Delta x$. Solving (4.10), we find $E^{(3)} \sim \eta\, \frac{\partial E}{\partial x} \sum_{n,m,p \text{ odd}} A_m A_p A_{n-m-p} \cdot e^{i\Omega_n t}$ on $[0, X_c + \Delta x)$. This immediately yields the leading order of the coefficient of the term in (4.12), which is cubic in the amplitudes

$$\text{(C.15)} \qquad \gamma_1 \sim \int_0^{X+\Delta x} \eta \frac{\partial E}{\partial x}\, dx \sim - \frac{L^3\, v_2'^2}{12\, v_1'^3\, (E_2 - E_1)^2\, J^2}.$$

Note that (C.13) yields a contribution $3\, v_1''\, (E_2 - E_1)\, J\, \gamma_1 / (v_2'\, L)$, which is typically negligible in comparison with $\gamma_1$. Our derivation has ignored other terms in $E^{(1)}$ and $E^{(2)}$ which provide contributions to the integral (4.12) of the same order as (C.13) and are therefore subdominant compared to (C.15) when $v_2'$ and $(E_2 - E_1)$ are $O(1)$.

To calculate the coefficients of the nonlinear terms in the amplitude equations, we only need to know the coefficient of $\frac{\partial A_n}{\partial T}$ in the nonresonance condition (3.4). By inspecting (4.10)–(4.12) and taking into account (2.29) and (2.30), we find that the largest contribution to that coefficient is

$$\text{(C.16)} \qquad \frac{X_c\, L}{J\, v_1'}\, \frac{\partial A_n}{\partial T}.$$

Note that the correction (C.2) to (4.12) divided by the coefficient of $\frac{\partial A_n}{\partial T}$ in (C.16) yields the correct coefficient of $\Omega_n^2$ in the dispersion relation (2.24). The other linear term is computed straightforwardly. Together with the dispersion relation (2.24), (C.15) and (C.16) provide the leading-order approximation to the amplitude equations (5.4) and (5.5) of § 5.

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## REFERENCES

[1] K. Aoki and K. Yamamoto, *Firing wave instability of the current filaments in a semiconductor: An analogy with neurodynamics*, Phys. Lett. A, 98 (1983), pp. 72–78.

[2] L. L. Bonilla, *Solitary waves in semiconductors with finite geometry and the Gunn effect*, SIAM J. Appl. Math., 51 (1991) pp. 727–747.

[3] ———, *Small signal analysis of spontaneous current instabilities in extrinsic semiconductors with trapping: Application to p-type ultrapure germanium*, Phys. Rev. B, 45 (1992), pp. 1142–1154.

[4] ———, *Theory of solitary waves and spontaneous current instabilities in dc voltage biased extrinsic semiconductors*, Phys. D, 55 (1992), pp. 182–196.

[5] ———, *Dynamics of electric field domains in superlattices*, in Nonlinear Dynamics and Pattern Formation in Semiconductors and Devices, ed. F.-J. Niedernostheide, Springer-Verlag, Berlin, 1995, pp. 1–20.
L. L. Bonilla, J. G. Galán, J. A. Cuesta, F. C. Martínez, and J. M. Molera, *Dynamics of electric field domains and oscillations of the photocurrent in a simple superlattice model*, Phys. Rev. B, 50 (1994), pp. 8644–8657.

[6] L. L. Bonilla and F. J. Higuera, *Gunn instability in finite samples of GaAs.* I. *Stationary states, stability and boundary conditions*, Phys. D, 52 (1991), pp. 458–476.

[7] L. L. Bonilla, F. J. Higuera, and J. M. Vega, *Stability of stationary solutions of extended reaction-diffusion-convection equations on a finite segment*, Appl. Math. Lett., 4 (1991), pp. 41–44.

[8] L. L. Bonilla, F. J. Higuera and S. Venakides, *The Gunn effect: instability of the steady state and stability of the solitary wave in long extrinsic semiconductors*, SIAM J. Appl. Math., 54 (1994), pp. 1521–1541.

[9] L. L. Bonilla, J. C. Neu, and R. Spigler, *Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators*, J. Statist. Phys., 67 (1992), pp. 313–330.

[10] L. L. Bonilla and S. W. Teitsworth, *Theory of periodic and solitary space charge waves in extrinsic semiconductors*, Phys. D, 50 (1991), pp. 545–559.
L. L. Bonilla and J. M. Vega, *On the stability of wavefronts and solitary space charge waves in extrinsic semiconductors under current bias*, Phys. Lett. A, 156 (1991), pp. 179–182.

[11] P. N. Butcher, *The Gunn effect*, Rep. Progr. Phys., 30 (1967), pp. 97–148.

[12] M. Büttiker and H. Thomas, *Bifurcation and stability of dynamical structures at a current instability*, Z. Phys. B, 34 (1979), pp. 301–311.

[13] I. R. Cantalapiedra, L. L. Bonilla, M. J. Bergmann, and S. W. Teitsworth, *Solitary-wave dynamics in extrinsic semiconductors under dc voltage bias*, Phys. Rev. B, 48 (1993), pp. 12278–12281.
L. L. Bonilla, I. R. Cantalapiedra, M. J. Bergmann, and S. W. Teitsworth, *Onset of current oscillations in extrinsic semiconductors under dc voltage bias*, Semicond. Sci. Technol., 9 (1994), pp. 599–602.

[14] M. C. Cross and P. C. Hohenberg, *Pattern formation outside of equilibrium*, Rev. Mod. Phys., 65 (1993), pp. 851–1112.

[15] A. Diaz-Guilera and J. M. Rubí, *On fluctuations about nonequilibrium steady states near Gunn instability*, Phys. A, 135 (1986), pp. 180–212.

[16] M. Golubitsky and W. F. Langford, *Classification of unfoldings of degenerate Hopf bifurcations*, J. Differential Equations, 41 (1981) , pp. 375–415.

[17] H. L. Grubin, *Dynamic cathode boundary fields and transferred electron oscillators*, IEEE Trans. Electron Devices, ED-23 (1976), pp. 1012–1015.

[18] H. L. GRUBIN, M. P. SHAW, AND P. R. SOLOMON, *On the form and stability of electric-field profiles within a negative differential mobility semiconductor*, IEEE Trans. Electron Devices, ED-20 (1973), pp. 63–78.

[19] J. B. GUNN, *Microwave oscillations in III-IV semiconductors*, Solid State Comm., 1 (1963), pp. 88–91.

[20] ———, *Instabilities of current and of potential distribution in GaAs and InP*, in Proc. Symposium on Plasma Effects in Solids, Dunod, Paris, 1965, pp. 199–207.

[21] ———, *Electronic transport properties relevant to instabilities in GaAs*, J. Phys. Soc. Japan (Suppl.), 21 (1966), pp. 505–507.

[22] F. J. HIGUERA AND L. L. BONILLA, *Gunn instability in finite samples of GaAs*. II. *Oscillatory states in long samples*, Phys. D, 57 (1992), pp. 161–184.

[23] S. KABASHIMA, H. YAMAZAKI, AND T. KAWAKUBO, *Critical fluctuation near threshold of Gunn instability*, J. Phys. Soc. Japan, 40 (1976), pp. 921–924.

[24] A. M. KAHN, D. J. MAR, AND R. M. WESTERVELT, *Moving space-charge domains in p-type ultrapure Germanium*. I. *Spatial measurements*, Phys. Rev. B, 43 (1991), pp. 9740–9749.

[25] ———, *Spatial measurements near the instability threshold in ultrapure germanium*, Phys. Rev. B, 45, (1992), pp. 8342–8347.

[26] ———, *Dynamics of space-charge domains in ultrapure Ge*, Phys. Rev. Lett., 68 (1992), pp. 369–372.

[27] J. KEIZER, *Calculation of voltage fluctuations at the Gunn instability*, J. Chem. Phys., 74 (1981), pp. 1350–1356.

[28] M. KINDELAN, L. L. BONILLA, AND F. J. HIGUERA, Onset of the Gunn effect in semiconductors: bifurcation analysis and numerical simulations. Preprint, 1995.

[29] B. W. KNIGHT AND G. A. PETERSON, *Nonlinear analysis of the Gunn effect*, Phys. Rev., 147 (1966), pp. 617-621.
———, *Theory of the Gunn effect*, Phys. Rev., 155 (1967), pp. 393–404.

[30] S. KOGELMAN AND J. B. KELLER, *Transient behavior of unstable nonlinear systems with applications to the Bénard and Taylor problems*, SIAM J. Appl. Math., 20 (1971), pp. 619–637.

[31] H. KROEMER, *Nonlinear space-charge domain dynamics in a semiconductor with negative differential mobility*, IEEE Trans. Electron Devices, ED-13 (1966), pp. 27–40.

[32] H. KROEMER, *The Gunn effect under imperfect cathode boundary conditions*, IEEE Trans. Electron Devices, ED-15 (1968), pp. 819–837.

[33] S.-H. KWOK, Novel electric field effects in GaAs-(Al, Ga)As superlattices, Ph.D. thesis, University of Michigan, Ann Arbor, 1994. See also S.-H. Kwok, T. C. Norris, L. L. Bonilla, J. Galán, J. A. Cuesta, F. C. Martínez, J. M. Molera, H. T. Grahn, K. Ploog and R. Merlin, *Domain wall kinetics and tunneling-induced instabilities in superlattices*, Phys. Rev. B, 51 (1995), pp. 10171–10174.

[34] J. LIANG, *On a nonlinear integrodifferential semiconductor model*, SIAM J. Math. Anal., 25 (1994), pp. 1375–1392.

[35] P. MANNEVILLE, *Dissipative Structures and Weak Turbulence*. Academic Press, Boston, 1990.

[36] P. A. MARKOWICH, C. A. RINGHOFER, AND C. SCHMEISER, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.

[37] D. E. MCCUMBER AND A. G. CHYNOWETH, *Theory of negative-conductance amplification and of Gunn instabilities in "two-valley" semiconductors*, IEEE Trans. Electron. Devices, ED-13 (1966), pp. 4–21.

[38] E. MOSEKILDE, R. FELDBERG, C. KNUDSEN, AND M. HIMDSHOLM, *Mode locking and spatiotemporal chaos in periodically driven Gunn diodes*, Phys. Rev. B, 41 (1990), pp. 2298–2306.
E. MOSEKILDE, J. S. THOMSEN, C. KNUDSEN, AND R. FELDBERG, *Phase diagrams for periodically driven Gunn diodes*, Phys. D, 66 (1993), pp. 145–153.

[39] K. NAKAMURA, *Statistical dynamics of the Gunn instability near threshold*, J. Phys. Soc. Japan, 38 (1975), pp. 46–50.

[40] B. K. RIDLEY, *Specific resistance in solids*, Proc. Phys. Soc., 82 (1963), pp. 954–966.

[41] D. G. SEILER, C. L. LITTLER, AND R. J. JUSTICE, *Nonlinear oscillations and chaos in n-InSb*, Phys. Lett. A, 108 (1985), pp. 462–467.

[42] M. P. SHAW, H. L. GRUBIN, AND P. R. SOLOMON, *The Gunn-Hilsum Effect*, Academic Press, New York, 1979.

[43] M. P. SHAW, V. V. MITIN, E. SCHÖLL, AND H. L. GRUBIN, *The Physics of Instabilities in Solid State Electron Devices*, Plenum, New York, 1992.

[44] S. M. SZE, Physics of Semiconductor Devices, 2nd ed., John Wiley, New York, 1981.

[45] P. SZMOLYAN, *Traveling waves in GaAs semiconductors*, Phys. D, 39 (1989), pp. 393–404.

[46] S. W. TEITSWORTH, *The physics of space charge instabilities and temporal chaos in extrinsic semiconductors*, Appl. Phys. A, 48 (1989), pp. 127–136.